Bounds for Green's functions on hyperbolic Riemann surfaces of finite volume

Dissertation

zur Erlangung des akademischen Grades
doctor rerum naturalium
(Dr. rer. nat.)
im Fach Mathematik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät II
der Humboldt-Universität zu Berlin

von

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eingereicht am: 6. Mai 2013
Tag der Verteidigung: 13. September 2013

To my mom and dad

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Berlin, den 6. Mai 2013 Anilatmaja Aryasomayajula

Abstract

Let X be a non-compact hyperbolic Riemann surface of finite hyperbolic volume with genus $g \geq 1$. By the uniformization theorem from complex analysis, X can be realized as the quotient space $\Gamma\backslash\mathbb{H}$, where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian subgroup of the first kind acting by fractional linear transformations on $\mathbb{H}$. Since X is non-compact, Γ admits parabolic elements. We assume that Γ does not admit torsion points.

Associated to the canonical volume form $\mu_{\mathrm{can}}(z)$ on X, there exists a canonical Green's function $g_{\mathrm{can}}(z,w)$ on $X \times X$ which is smooth away from the diagonal, and is log-singular along the diagonal. In this thesis, we obtain bounds for the canonical Green's function $g_{\mathrm{can}}(z,w)$ away from the diagonal, in terms of invariants coming from the hyperbolic geometry of X.

We define the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ on $X \times X$ via the heat kernel $K_{\mathrm{hyp}}(t;z,w)$ defined on $\mathbb{R}_{>0} \times X \times X$. We then study its behavior at the parabolic fixed points, and then proceed to express the difference $g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)$ in terms of integrals involving the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ and the canonical volume form $\mu_{\mathrm{can}}(z)$ for all $z, w \in X$.

We then prove a formula which expresses the canonical volume form $\mu_{\mathrm{can}}(z)$ in terms of the hyperbolic volume form $\mu_{\mathrm{hyp}}(z)$ and the hyperbolic heat kernel $K_{\mathrm{hyp}}(t;z,w)$. Using this relation, we derive an expression for the difference of the hyperbolic and canonical Green's functions $g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)$ solely in terms of expressions related to the hyperbolic heat kernel.

Using the existing bounds for the hyperbolic heat kernel $K_{\mathrm{hyp}}(t;z,w)$, we first derive upper bounds for the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$, and then for the difference $g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)$ in terms of invariants coming from the hyperbolic geometry of X. Using these estimates, we derive bounds for the canonical Green's function $g_{\mathrm{can}}(z,w)$, both away from the parabolic fixed points and at the parabolic fixed points.

Keywords:
Hyperbolic Green's function, Canonical Green's function

Abstrakt

Es sei X eine nicht-kompakte Riemannsche Fläche von endlichem hyperbolischen Volumen und Geschlecht $g \geq 1$. Gemäß des Uniformisierungssatzes aus der komplexen Analysis lässt sich X als Quotientenraum $\Gamma\backslash\mathbb{H}$ realisieren, wobei $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ eine Fuchssche Untergruppe erster Art ist, welche durch gebrochen-lineare Transformation auf $\mathbb{H}$ wirkt. Wegen der nicht-Kompaktheit von X enthält Γ parabolische Elemente. Wir nehmen an, dass Γ keine Torsionspunkte zulässt.

Es existiert eine kanonische Greensche Funktion $g_{\mathrm{can}}(z,w)$ auf $X \times X$ bezüglich der kanonischen Volumenform $\mu_{\mathrm{can}}(z)$ auf X, welche glatt außerhalb der Diagonale und log-singulär entlang der Diagonale ist. In der vorliegenden Arbeit bestimmen wir Schranken für die kanonische Greensche Funktion außerhalb der Diagonale in Termen von Invarianten aus der hyperbolischen Geometrie.

Wir definieren die hyperbolische Greensche Funktion $g_{\mathrm{hyp}}(z,w)$ auf $X \times X$ über den Wärmeleitungskern $K_{\mathrm{hyp}}(t;z,w)$ auf $\mathbb{R}_{>0} \times X \times X$. Danach untersuchen wir ihr Verhalten an den parabolischen Fixpunkten und beschreiben daran anschließend die Differenz $g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)$ für alle $z, w \in X$ mithilfe von Integralen, welche die hyperbolische Greensche Funktion $g_{\mathrm{hyp}}(z,w)$ sowie die kanonische Volumenform $\mu_{\mathrm{can}}(z)$ beinhalten.

Dann beweisen wir eine Formel, welche die kanonische Volumenform $\mu_{\mathrm{can}}(z)$ in Termen der hyperbolischen Volumenform $\mu_{\mathrm{hyp}}(z)$ und des hyperbolischen Wärmeleitungskerns $K_{\mathrm{hyp}}(t;z,w)$ ausdrückt. Damit gelingt es uns, einen Ausdruck für die Differenz der hyperbolischen und kanonischen Greenschen Funktion $g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)$ zu finden, und zwar ausschließlich in Termen, welche im Zusammenhang mit dem hyperbolischen Wärmeleitungskern stehen.

Unter Verwendung bereits existierender Schranken für $K_{\mathrm{hyp}}(t;z,w)$ erhalten wir obere Schranken für die hyperbolische Greensche Funktion $g_{\mathrm{hyp}}(z,w)$, und damit auch für die Differenz $g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)$, in Termen von Invarianten aus der hyperbolischen Geometrie. Mithilfe dieser Abschätzungen können wir die kanonische Greensche Funktion $g_{\mathrm{can}}(z,w)$ ausserhalb sowie an den parabolischen Fixpunkten von oben beschränken.

Schlüsselwörter:
Hyperbolische Greensche Funktion, Kanonische Greensche Funktion

Contents

Introduction

Background

In [2], Arakelov defined an intersection theory for divisors on an arithmetic surface by incorporating the associated compact Riemann surface with its complex analytic geometry. The contribution at infinity is calculated by using certain Green's functions defined on the corresponding Riemann surfaces. These Green's functions are known as the canonical Green's functions.

In [5], B. Edixhoven et al. devised an algorithm which for a given prime ℓ, computes the Galois representations modulo ℓ associated to a fixed modular form of arbitrary weight, in time polynomial in ℓ. To show that the complexity of the algorithm is polynomial in ℓ, they needed an upper bound for the canonical Green's function associated to the modular curve $X_1(\ell)$, as a function of ℓ. In [5], F. Merkl has derived an estimate of the canonical Green's function that is polynomial in ℓ, which proved sufficient to conclude that the algorithm has complexity that is polynomial in ℓ.

In [11], motivated by the work of B. Edixhoven, using completely different techniques, J. Jorgenson and J. Kramer derived estimates of the canonical Green's function on a compact hyperbolic Riemann surface, after removing its log-singularity along the diagonal. These estimates were derived in terms of invariants coming from hyperbolic geometry. As an application, they deduced bounds for the canonical Green's functions through covers and for families of modular curves. Their estimates of the canonical Green's function associated to the modular curve $X_1(\ell)$, as a function of ℓ are uniform in ℓ, which are much sharper than the one deduced by F. Merkl.

The main goal of the current thesis is to extend the methods of J. Jorgenson and J. Kramer from [11] to non-compact hyperbolic Riemann surfaces of finite hyperbolic volume. In this thesis, we derive estimates of the canonical Green's function on a non-compact hyperbolic Riemann surface of finite hyperbolic volume, after removing its log-singularity along the diagonal. Following the same techniques as in [11], we derive bounds for the canonical Green's functions through covers and for families of modular curves.

Estimates of the canonical Green's function at the parabolic fixed points are essential for calculating the Faltings height of a modular curve. In [1], in course of bounding the arithmetic self-intersection number of the relative dualizing sheaf for the modular curve $X_0(N)$, A. Abbes and E. Ullmo have obtained an estimate of the canonical Green's function when evaluated at two different parabolic fixed points. In [18], H. Mayer has extended the work of A. Abbes and E. Ullmo to the modular curve $X_1(N)$, and also computed an estimate of the canonical Green's function when evaluated at two different parabolic fixed points.

Motivated by these results, as an application of the analysis derived in this thesis, we compute an upper bound for the canonical Green's function when evaluated at two different parabolic fixed points.

Though we assume that our Riemann surface is devoid of torsion points, our methods easily extend to the case when the Riemann surface does admit torsion points. We hope to address the case of torsion points in a future article.

Notations

Let X be a non-compact hyperbolic Riemann surface of finite hyperbolic volume $\mathrm{vol}_{\mathrm{hyp}}(X)$ with genus $g \geq 1$. Then by the uniformization theorem from complex analysis, X can be realized as the quotient space $\Gamma\backslash\mathbb{H}$, where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian subgroup of the first kind acting via fractional linear transformations on the upper half-plane $\mathbb{H}$. We identify points on X with their pre-images in $\mathbb{H}$.

Let $\mathcal{P}$ denote the set of parabolic fixed points of Γ. We assume that Γ does not have torsion points. Let $\overline{X}$ denote the compactification of X obtained by adding the set of parabolic fixed points $\mathcal{P}$ to X, i.e., $\overline{X} = X \cup \mathcal{P}$.

Let Δ_{hyp} denote the hyperbolic Laplacian on X. Let $\mu_{\mathrm{hyp}}(z)$ denote the natural metric on X, which is compatible with its complex structure. Locally, for $z \in X$, it is given by

$$\mu_{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{dz \wedge d\overline{z}}{\mathrm{Im}(z)^2}.$$

The rescaled hyperbolic metric

$$\mu_{\mathrm{shyp}}(z) = \frac{\mu_{\mathrm{hyp}}(z)}{\mathrm{vol}_{\mathrm{hyp}}(X)}$$

measures the volume of X to be one.

Let $S_k(\Gamma)$ denote the $\mathbb{C}$-vector space of cusp forms of weight k with respect to Γ equipped with the Petersson inner product. Let $\{f_1, \ldots, f_g\}$ denote an orthonormal basis of $S_2(\Gamma)$ with respect to the Petersson inner product. The canonical metric $\mu_{\mathrm{can}}(z)$ is given by

$$\mu_{\mathrm{can}}(z) = \frac{i}{2g} \sum_{j=1}^{g} \left|f_j(z)\right|^2 dz \wedge d\overline{z}.$$

For $z, w \in X$, the canonical Green's function $g_{\mathrm{can}}(z,w)$ is defined as the solution of the differential equation

$$d_z d_z^c g_{\mathrm{can}}(z,w) + \delta_w(z) = \mu_{\mathrm{can}}(z),$$

with the normalization condition

$$\int_X g_{\mathrm{can}}(z,w)\, \mu_{\mathrm{can}}(z) = 0.$$

On the diagonal, $g_{\mathrm{can}}(z,w)$ admits a log-singularity, i.e., for $z, w \in X$, it satisfies

$$\lim_{w \to z} \left(g_{\mathrm{can}}(z,w) + \log \left| \vartheta_z(w) \right|^2 \right) = O_z(1), \tag{1}$$

where $\vartheta_z(w)$ denotes the local coordinate function for an open coordinate disk around the point z, and the contribution from the term $O_z(1)$ is a smooth function in z.

Let $K_{\mathrm{hyp}}(t; z, w)$ denote the hyperbolic heat kernel on $\mathbb{R}_{>0} \times X \times X$. To simplify notation, when $z = w$ we write $K_{\mathrm{hyp}}(t; z)$ instead of $K_{\mathrm{hyp}}(t; z, z)$. For $z, w \in X$ and $z \neq w$, the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ is defined as

$$g_{\mathrm{hyp}}(z,w) = 4\pi \int_0^\infty \left(K_{\mathrm{hyp}}(t; z, w) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) dt.$$

In analogy to the canonical Green's function $g_{\mathrm{can}}(z,w)$, the hyperbolic Green's function satisfies the differential equation

$$d_z d_z^c g_{\mathrm{hyp}}(z,w) + \delta_w(z) = \mu_{\mathrm{shyp}}(z),$$

with the normalization condition

$$\int_X g_{\mathrm{hyp}}(z,w)\, \mu_{\mathrm{hyp}}(z) = 0.$$

On the diagonal, $g_{\mathrm{hyp}}(z,w)$ admits a log-singularity, i.e., for $z, w \in X$, it satisfies

$$\lim_{w \to z} \left(g_{\mathrm{hyp}}(z,w) + \log \left| \vartheta_z(w) \right|^2 \right) = O_z(1),$$

where $\vartheta_z(w)$ and $O_z(1)$ are as in equation (1).

Main Results

We now summarize the main results of this thesis. We first generalize the result of J. Jorgenson and J. Kramer from [11]

$$g\, \mu_{\mathrm{can}}(z) = \\ \left(\frac{1}{4\pi} + \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) \mu_{\mathrm{hyp}}(z) + \frac{1}{2} \left(\int_0^\infty \Delta_{\mathrm{hyp}}\, K_{\mathrm{hyp}}(t; z) dt \right) \mu_{\mathrm{hyp}}(z), \tag{2}$$

which relates the hyperbolic and canonical volume forms to a relation of currents acting on smooth functions on $\overline{X}$ (see Theorem 2.9.5). We then extend this relation of currents to a certain class of singular functions (see Theorem 3.2.4). Noting that the hyperbolic Green's function belongs to this class of singular functions, we derive (see Corollary 3.2.7)

$$\begin{aligned} & g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \\ & \frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)\left(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;\zeta)dt\right)\mu_{\mathrm{hyp}}(\zeta)+ \\ & \frac{1}{2g}\int_X g_{\mathrm{hyp}}(w,\zeta)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{4g^2}, \end{aligned} \tag{3}$$

where C_{hyp} is a suitable constant.

The expression derived in equation (3) allows us to estimate the canonical Green's function $g_{\mathrm{can}}(z,w)$ solely in terms of invariants coming from the hyperbolic geometry of X.

Adapting the bounds for heat kernels from [11] to compact subsets of X, we compute an upper bound for the difference $g_{\mathrm{hyp}}(z,w)-g_{\mathrm{can}}(z,w)$ on a compact subset of X (see Theorem 6.1.12). The upper bound is expressed in terms of the first non-zero eigenvalue of the hyperbolic Laplacian on X, the injectivity radius of the compact subset, and other data coming from the hyperbolic geometry of X.

Using a result of P. Bruin from [4], we extend this upper bound to neighborhoods of parabolic fixed points. Furthermore, we derive two different upper bounds for the canonical Green's function, when evaluated at two different parabolic fixed points (see Corollary 6.2.9 and Theorem 7.1.14).

We then extend study these bounds through covers and for families of modular curves, which we now explain. Let $\mathcal{N} \subseteq \mathbb{N}$ be such that the modular curve $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ has genus $g_N \geq 1$, and $\Gamma_0(N)$ has no torsion points for $N \in \mathcal{N}$. Let $q_{\mathcal{N}}$ denote the smallest prime in $\mathcal{N}$. We denote the set of parabolic fixed points of $\Gamma_0(N)$ by $\mathcal{P}_N$ and its cardinality by $|\mathcal{P}_N|$.

Let $0 < \varepsilon < 1$ be any number such that for all $N \in \mathcal{N}$, the following condition holds true:

$$U_{N,\varepsilon}(p) \cap U_{N,\varepsilon}(q) = \emptyset \tag{4}$$

for all parabolic fixed points $p, q \in \mathcal{P}_N$ and $p \neq q$, where $U_{N,\varepsilon}(p)$, $U_{N,\varepsilon}(q)$ denote open coordinate disks of radius ε around $p, q \in \mathcal{P}_N$, respectively.
For a fixed $0 < \varepsilon < 1$ satisfying (4), put

$$Y_0(N)_\varepsilon = Y_0(N)\backslash \bigcup_{p\in\mathcal{P}_N} U_{N,\varepsilon}(p).$$

For $N \in \mathcal{N}$, let $g_{N,\mathrm{can}}(z,w)$ and $g_{N,\mathrm{hyp}}(z,w)$ denote the canonical and hyperbolic Green's functions defined on $Y_0(N) \times Y_0(N)$, respectively. Then, for

$N \in \mathcal{N}$ sufficiently large, we derive the estimate (see Theorem 7.2.13)

$$\sup_{z,w \in Y_0(N)_\varepsilon} \Big| g_{N,\mathrm{hyp}}(z,w) - g_{N,\mathrm{can}}(z,w) \Big| = O_{q_{\mathcal{N}},\varepsilon}\bigg(g_N \, |\mathcal{P}_N| \bigg(1 + \frac{1}{\lambda_{N,1}} \bigg) \bigg), \quad (5)$$

where $\lambda_{N,1}$ denotes the first non-zero eigenvalue of the hyperbolic Laplacian Δ_{hyp} on $Y_0(N)$. Let $p, q \in \mathcal{P}_N$ be two parabolic fixed points with $p \neq q$. Then, for $N \in \mathcal{N}$ sufficiently large, we prove that (see Corollary 7.2.17)

$$\big| g_{N,\mathrm{can}}(p,q) \big| = O_{q_{\mathcal{N}},\varepsilon}\bigg(g_N \, |\mathcal{P}_N| \bigg(1 + \frac{1}{\lambda_{N,1}} \bigg) \bigg). \quad (6)$$

These results extend with notational changes to other families of modular curves like $\{Y_1(N)\}_{N \in \mathcal{N}}$ and $\{Y(N)\}_{N \in \mathcal{N}}$.

Unlike the estimates derived in [11], from (5) and (6), it is easy to see that our estimates are not uniformly bounded in N. The initial aim of the thesis was to derive estimates similar to the ones obtained in [11]. Although our estimates are not optimal, it is quite feasible to extend our methods to achieve the optimal estimates. For this we need to revisit an estimate (see Theorem 5.2.11), which we have directly adapted to our situation from [11].

It is to be mentioned that P. Bruin in his doctoral thesis [4], in course of generalizing Edixhoven's algorithm has derived bounds for canonical Green's functions on non-compact hyperbolic Riemann surfaces of finite hyperbolic volume. His bounds are slightly stronger than ours.

Outline

In Chapter 1, we introduce the basic notions. We introduce the main players, namely, the canonical Green's function, the hyperbolic heat kernel, and the hyperbolic Green's function defined on a non-compact hyperbolic Riemann surface of finite hyperbolic volume. We state their well-known properties, and explain how they are related to the more extensively studied Green's functions like the free-space Green's function and the automorphic Green's function.

In Chapter 2, we start with investigating the behavior of the hyperbolic Green's function at the parabolic fixed points, and proceed to show that it defines a Green's current. We then extend the key identity (2) to torsion and parabolic fixed points at the level of currents.

In Chapter 3, we extend the key identity from Chapter 2 to a certain class of singular functions. Noting that the hyperbolic Green's function belongs to this class of singular functions and using the extended version of the key identity, we prove equation (3).

In Chapter 4, we introduce certain automorphic functions, and compute their asymptotics at the parabolic fixed points. We then show that the right-hand side of equation (3) can be further decomposed into integrals involving these automorphic functions and the hyperbolic volume form.

In Chapter 5, we introduce certain hyperbolic-geometric invariants associated to a compact subset of the Riemann surface. In [11] upper bounds for the hyperbolic heat kernel and the hyperbolic Green's function were derived in terms of these hyperbolic-geometric invariants. We adapt the upper bounds from [11] to a compact subset of the Riemann surface, and proceed to extend these upper bounds to the neighborhoods of parabolic fixed points.

In Chapter 6, using the above mentioned decomposition from Chapter 4 and using the upper bounds derived in Chapter 5 for the hyperbolic heat kernel and the hyperbolic Green's function, we compute upper bounds for the canonical Green's function on a compact subset of the Riemann surface, after removing its log-singularity along the diagonal. We then extend these upper bounds to the neighborhoods of parabolic fixed points.

In Chapter 7, using the asymptotics of certain automorphic functions from Chapter 4, we compute an upper bound for the canonical Green's function when evaluated at two different parabolic fixed points. Furthermore, using the upper bounds derived in Chapter 6, we compute upper bounds for the canonical Green's functions through covers and for families of modular curves.

Chapter 1

Background material

In this chapter we set up the notation for the rest of the thesis.

In Section 1.1, we describe the structure of the Riemann surface associated to the quotient space $\Gamma\backslash\mathbb{H}$, where $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H}$.

In Section 1.2, we introduce the hyperbolic metric and the hyperbolic Laplacian on the Riemann surface $\Gamma\backslash\mathbb{H}$.

In Sections 1.3 and 1.4, we describe the canonical metric and the canonical Green's function defined on the Riemann surface $\Gamma\backslash\mathbb{H}$, respectively.

In Section 1.5, we recall the notions of the parabolic Eisenstein series and the Kronecker's limit function associated to a parabolic fixed point, and describe their Fourier expansions at the parabolic fixed points.

In Section 1.6, we introduce the Hilbert space of square integrable functions with respect to the hyperbolic metric.

In Section 1.7, we introduce the hyperbolic heat kernels defined on $\mathbb{H}$, and the quotient space $\Gamma\backslash\mathbb{H}$. We then state the spectral expansion of the hyperbolic heat kernel defined on $\Gamma\backslash\mathbb{H}$, and proceed to describe its long-time and short-time asymptotics.

In Sections 1.8, 1.9, and 1.10, we introduce the free-space Green's function, the automorphic Green's function, and the hyperbolic Green's function, respectively. We state the well-known properties of these Green's functions, and show how they are related to the heat kernels.

In Section 1.11, we recall a key identity which was first proved in [11] for a compact quotient $\Gamma\backslash\mathbb{H}$. Using this identity, estimates of the canonical Green's function were obtained in [11]. In the coming chapters, we extend this identity to torsion and parabolic fixed points.

1.1 Structure of $\overline{X}$ as a compact Riemann surface

Let $\mathbb{C}$ denote the complex plane. For $z \in \mathbb{C}$, let $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ denote the real and imaginary parts of z, respectively. Let

$$\mathbb{H} = \{z \in \mathbb{C} \,|\, y = \mathrm{Im}(z) > 0\}$$

be the upper half-plane. Let $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind acting by fractional linear transformations on $\mathbb{H}$. Let X be the quotient space $\Gamma\backslash\mathbb{H}$ of genus $g \geq 1$. The quotient space X admits the structure of a Riemann surface.

Let $\mathcal{T}$, $\mathcal{P}$ be the set of torsion points, parabolic fixed points of X, respectively, and $|\mathcal{P}|$ denote the number of parabolic fixed points; put $\mathcal{S} = \mathcal{T} \cup \mathcal{P}$. Since Γ is a Fuchsian subgroup of the first kind, X admits only finitely many torsion points and parabolic fixed points. For $t \in \mathcal{T}$, let m_t denote the order of t; for $p \in \mathcal{P}$, put $m_p = \infty$; for $z \in X\backslash\mathcal{T}$, put $m_z = 1$.

Let $\mathbb{H}^*$ denote $\mathbb{H} \cup \mathbb{P}_\Gamma$, where $\mathbb{P}_\Gamma$ is a suitable denumerable subset of $\mathbb{P}^1(\mathbb{R})$, and let $\overline{X}$ denote the quotient space $\Gamma\backslash\mathbb{H}^*$; we have $\overline{X} = X \cup \mathcal{P}$.

Locally, away from the torsion points and the parabolic fixed points, we identify $\overline{X}$ with its universal cover $\mathbb{H}$, and hence, denote the points on $\overline{X}\backslash\mathcal{S}$ by the same letter as the points on $\mathbb{H}$.

The quotient space $\overline{X}$ admits the structure of a compact Riemann surface. $\overline{X}$ can be viewed as the compactification of X, obtained by adding the set of parabolic fixed points $\mathcal{P}$ to X. We refer the reader to Section 1.8 in [19], for the details regarding the structure of $\overline{X}$ as a compact Riemann surface.

We now describe the coordinate neighborhoods and local coordinate functions of the torsion and parabolic fixed points of $\overline{X}$. For $z \in \overline{X}$, let $U_r(z)$ denote an open coordinate disk of radius r around z. Let us denote the coordinate function for $w \in U_r(z)$ by $\vartheta_z(w)$.

For $z \in \overline{X}\backslash\mathcal{S}$, and $w \in U_r(z)$, the local coordinate function $\vartheta_z(w)$ is given by

$$\vartheta_z(w) = w - z.$$

Let $z = t \in \mathcal{T}$ be a torsion point, and $w \in U_r(t)$, then $\vartheta_t(w)$ is given by

$$\vartheta_t(w) = \left(\frac{w-t}{w-\overline{t}}\right)^{m_t}.$$

Let $z = p \in \mathcal{P}$ be a parabolic fixed point. So there exists a scaling matrix $\sigma_p \in \mathrm{PSL}_2(\mathbb{R})$ satisfying the relations

$$\sigma_p i\infty = p \qquad \text{and} \qquad \sigma_p^{-1}\Gamma_p\sigma_p = \langle\gamma_\infty\rangle, \tag{1.1}$$

where

$$\gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \Gamma_p = \langle\gamma_p\rangle, \tag{1.2}$$

denotes the stabilizer of p, with generator γ_p. Then for $w \in U_r(p)$, $\vartheta_p(w)$ is given by

$$\vartheta_p(w) = e^{2\pi i \sigma_p^{-1} w}.$$

1.2 Hyperbolic metric

Definition 1.2.1. We denote the (1,1)-form corresponding to the hyperbolic metric of X, which is compatible with the complex structure on X and has constant negative curvature equal to minus one, by $\mu_{\mathrm{hyp}}(z)$. Locally, for $z \in X \backslash \mathcal{T}$, it is given by

$$\mu_{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{dz \wedge d\overline{z}}{\mathrm{Im}(z)^2}.$$

In the neighborhood of a torsion point $t \in \mathcal{T}$, we see that the hyperbolic metric can be written using local coordinates $\vartheta_t(z)$ as

$$\mu_{\mathrm{hyp}}(z) = \frac{2i}{m_t^2} \cdot \frac{d\vartheta_t \wedge d\overline{\vartheta}_t}{|\vartheta_t|^{2(1-1/m_t)} \big(1 - |\vartheta_t|^{2/m_t}\big)^2}. \tag{1.3}$$

From equation (1.3), it follows that though the hyperbolic metric is singular at torsion points, it still remains integrable at these points.

Similarly, in the neighborhood of a parabolic fixed point $p \in \mathcal{P}$, we find that the hyperbolic metric can be expressed in local coordinates as

$$\mu_{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{d\vartheta_p \wedge d\overline{\vartheta}_p}{\big(|\vartheta_p| \log |\vartheta_p|\big)^2}. \tag{1.4}$$

Let $\mathrm{vol}_{\mathrm{hyp}}(X)$ be the volume of X with respect to the hyperbolic metric μ_{hyp}. It is given by the formula

$$\mathrm{vol}_{\mathrm{hyp}}(X) = 2\pi \Bigg(2g - 2 + |\mathcal{P}| + \sum_{t \in T} \bigg(1 - \frac{1}{m_t} \bigg) \Bigg).$$

The rescaled hyperbolic metric

$$\mu_{\mathrm{shyp}}(z) = \frac{\mu_{\mathrm{hyp}}(z)}{\mathrm{vol}_{\mathrm{hyp}}(X)},$$

measures the volume of X to be one.

Definition 1.2.2. Locally, for $z \in X$, the hyperbolic Laplacian Δ_{hyp} on X is given by

$$\Delta_{\mathrm{hyp}} = -y^2 \bigg(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \bigg) = -4y^2 \bigg(\frac{\partial^2}{\partial z \partial \overline{z}} \bigg).$$

Recall that $d = \big(\partial + \overline{\partial}\big)$, $d^c = \dfrac{1}{4\pi i}\big(\partial - \overline{\partial}\big)$, and $dd^c = -\dfrac{\partial\overline{\partial}}{2\pi i}$. So for any smooth function f on X, we find

$$-4\pi dd^c f = \left(\Delta_{\mathrm{hyp}} f\right) \mu_{\mathrm{hyp}}\,.$$

Since X admits parabolic fixed points, Δ_{hyp} admits both a discrete and a continuous spectrum.

1.3 Canonical metric

Let $S_k(\Gamma)$ denote the $\mathbb{C}$-vector space of cusp forms of weight k with respect to Γ equipped with the Petersson inner product

$$\langle f, g\rangle = \frac{i}{2}\int_X f(z)\overline{g(z)}\,\mathrm{Im}(z)^k\cdot\frac{dz\wedge d\overline{z}}{\mathrm{Im}(z)^2}\qquad\big(\text{where } f,g\in S_k(\Gamma)\big).$$

Definition 1.3.1. Let $\{f_1,\ldots,f_g\}$ denote an orthonormal basis of $S_2(\Gamma)$ with respect to the Petersson inner product. Then, the (1,1)-form $\mu_{\mathrm{can}}(z)$ corresponding to the canonical metric of X is given by

$$\mu_{\mathrm{can}}(z) = \frac{i}{2g}\sum_{j=1}^{g}|f_j(z)|^2\,dz\wedge d\overline{z}.$$

The canonical metric $\mu_{\mathrm{can}}(z)$ remains smooth at the torsion and parabolic fixed points, and measures the volume of X to be one.

Definition 1.3.2. Put

$$d_X = \sup_{z\in X}\frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)}. \tag{1.5}$$

Since the (1,1)-form $\mu_{\mathrm{can}}(z)$ remains smooth at the torsion and parabolic fixed points, and $1/\,\mu_{\mathrm{shyp}}(z)$ is zero at these points, the quantity d_X is well defined.

1.4 Canonical Green's function

Definition 1.4.1. For $z, w\in X$, the canonical Green's function $g_{\mathrm{can}}(z,w)$ is defined as the solution of the differential equation

$$d_z d_z^c g_{\mathrm{can}}(z,w) + \delta_w(z) = \mu_{\mathrm{can}}(z), \tag{1.6}$$

with the normalization condition

$$\int_X g_{\mathrm{can}}(z,w)\,\mu_{\mathrm{can}}(z) = 0. \tag{1.7}$$

The canonical Green's function $g_{\mathrm{can}}(z,w)$ admits a log-singularity at $z = w$, i.e., for $z,w\in X$, it satisfies

$$\lim_{w\to z}\big(g_{\mathrm{can}}(z,w) + \log|\vartheta_z(w)|^2\big) = O_z(1).$$

We refer the reader to Section 2.2 for the details regarding the existence, uniqueness, and symmetry of the canonical Green's function.

1.5 Parabolic Eisenstein series

In this section, we introduce the parabolic Eisenstein series and the Kronecker's limit function associated to a parabolic fixed point. We also describe the Fourier expansions of these two functions at the parabolic fixed points.

Definition 1.5.1. For $z \in X$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},p}(z,s)$ corresponding to a parabolic fixed point $p \in \mathcal{P}$ is defined by the series

$$\mathcal{E}_{\mathrm{par},p}(z,s) = \sum_{\gamma \in \Gamma_p \backslash \Gamma} \mathrm{Im}(\sigma_p^{-1}\gamma z)^s,$$

where Γ_p and σ_p are as in Section 1.1.

The following theorem gives the Laurent expansion of the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},p}(z,s)$ associated to a parabolic fixed point $p \in \mathcal{P}$ at $s = 1$.

Theorem 1.5.2. *For $z \in X$, the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},p}(z,s)$ associated to a parabolic fixed point $p \in \mathcal{P}$ converges absolutely and uniformly for* $\mathrm{Re}(s) > 1$. *It admits a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = 1$, and the Laurent expansion at $s = 1$ is of the form*

$$\mathcal{E}_{\mathrm{par},p}(z,s) = \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \cdot \frac{1}{s-1} + \kappa_p(z) + O_z(s-1), \tag{1.8}$$

where $\kappa_p(z)$ the constant term of $\mathcal{E}_{\mathrm{par},p}(z,s)$ at $s = 1$ is called Kronecker's limit function.

Proof. The proof can be read from chapter 6 of [8]. □

The following theorem describes the Fourier expansion of the Kronecker's limit function $\kappa_p(z)$ associated to a parabolic fixed point $p \in \mathcal{P}$ at the parabolic fixed points.

Theorem 1.5.3. *For $z \in X$, and $p, q \in \mathcal{P}$, the Kronecker's limit function $\kappa_p(\sigma_q z)$ admits a Fourier expansion of the form*

$$\kappa_p(\sigma_q z) =$$
$$\sum_{n<0} k_{p,q}(n) e^{2\pi i n \bar{z}} + \delta_{p,q} \mathrm{Im}(z) + k_{p,q}(0) - \frac{\log\big(\mathrm{Im}(z)\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)} + \sum_{n>0} k_{p,q}(n) e^{2\pi i n z},$$

with Fourier coefficients $k_{p,q}(n) \in \mathbb{C}$.

Proof. We refer the reader to Theorem 1.1 of [16] for the proof. □

Corollary 1.5.4. *For $p, q \in \mathcal{P}$, as $z \in X$ approaches q, we have*

$$\begin{aligned}
\kappa_p(z) &= \delta_{p,q} \mathrm{Im}(\sigma_q^{-1} z) - \frac{\log\big(\mathrm{Im}(\sigma_q^{-1} z)\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)} + O_z(1) \\
&= \delta_{p,q}\Big(- \frac{\log|\vartheta_q(z)|}{2\pi}\Big) - \frac{\log\big(- \log|\vartheta_q(z)|\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)} + O_z(1),
\end{aligned}$$

where the contribution from the term $O_z(1)$ in the above equation is a smooth function in z.

Proof. The corollary follows easily from Theorem 1.5.3. $\square$

The following theorem describes the Fourier expansion of the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},p}(z,s)$ associated to a parabolic fixed point $p \in \mathcal{P}$ at the parabolic fixed points.

Theorem 1.5.5. *Let $p, q \in \mathcal{P}$, then for $z \in X$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},p}(\sigma_q z, s)$ associated to $p \in \mathcal{P}$, admits a Fourier expansion of the form*

$$\mathcal{E}_{\mathrm{par},p}(\sigma_q z, s) = \delta_{p,q} y^s + \alpha_{p,q}(s) y^{1-s} + \sum_{n \neq 0} \alpha_{p,q}(n,s) W_s(nz),$$

where $\alpha_{p,q}(s)$ and $\alpha_{p,q}(n,s)$ are given by equations (3.21) and (3.22) in [8], respectively, and $W_s(nz)$ is the Whittaker function given by equation (A.6).

Proof. We refer the reader to Theorem 3.4 in [8]. $\square$

Remark 1.5.6. For $z \in X$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, from the definition of the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},p}(z,s)$ associated to the parabolic fixed point $p \in \mathcal{P}$, it follows that

$$\overline{\mathcal{E}_{\mathrm{par},p}(z,\overline{s})} = \mathcal{E}_{\mathrm{par},p}(z,s).$$

Using the above relation and the Fourier expansion of the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},p}(z,s)$ stated above in Theorem 1.5.5, and from the definitions of the Fourier coefficients $\alpha_{p,q}(s)$ and $\alpha_{p,q}(n,s)$, we derive

$$\mathcal{E}_{\mathrm{par},p}(\sigma_q z, s) = \delta_{p,q} y^s + \alpha_{p,q}(s) y^{1-s} + \sum_{n \neq 0} \alpha_{p,q}(n,s) \overline{W_{\overline{s}}(nz)}. \tag{1.9}$$

The above equation will come handy in Chapter 7.

Corollary 1.5.7. *For $p, q \in \mathcal{P}$, as $z \in X$ approaches q, we have*

$$\begin{aligned}\mathcal{E}_{\mathrm{par},p}(z,s) = \delta_{p,q} \operatorname{Im}(\sigma_q^{-1} z)^s + \alpha_{p,q}(s) \operatorname{Im}(\sigma_q^{-1} z)^{1-s} + \\ O\Big(\big(1 + \operatorname{Im}(\sigma_q^{-1} z)^{-\mathrm{Re}(s)}\big) e^{-2\pi \operatorname{Im}(\sigma_q^{-1} z)}\Big).\end{aligned}$$

Proof. We refer the reader to Corollary 3.5 in [8]. $\square$

1.6 Space of square integrable functions $L^2(X)$

Definition 1.6.1. Let $L^2(X)$ denote the space of square integrable functions on X with respect to the hyperbolic (1,1)-form $\mu_{\mathrm{hyp}}(z)$, i.e., every $f \in L^2(X)$ satisfies the condition

$$\|f\|^2 = \int_X |f(z)|^2 \, \mu_{\mathrm{hyp}}(z) < \infty.$$

Definition 1.6.2. There exists a natural inner product $\langle \cdot, \cdot \rangle$ on $L^2(X)$ given by

$$\langle f, g \rangle = \int_X f(z)\overline{g(z)}\, \mu_{\mathrm{hyp}}(z),$$

where $f, g \in L^2(X)$, making $L^2(X)$ into a Hilbert space.

Theorem 1.6.3. *Every $f \in L^2(X)$ admits the spectral expansion*

$$f(z) = \sum_{n=0}^{\infty} \langle f, \varphi_n(z) \rangle \varphi_n(z) + \\ \frac{1}{4\pi} \sum_{p \in \mathcal{P}} \int_{-\infty}^{\infty} \langle f, \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) \rangle \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) dr,$$

where $\{\varphi_n(z)\}$ denotes the set of orthonormal eigenfunctions for the discrete spectrum of Δ_{hyp}, and $\{\mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir)\}$ denotes the set of eigenfunctions for the continuous spectrum of Δ_{hyp}, with $\mathcal{E}_{\mathrm{par},p}(z, s)$ denoting the parabolic Eisenstein series for the parabolic fixed point $p \in \mathcal{P}$.

Proof. We refer the reader to Theorem 7.3 in [8] for the proof. □

Proposition 1.6.4. *Let $f, g \in L^2(X)$ admitting the following spectral expansions*

$$f(z) = \sum_{n=0}^{\infty} f_n \varphi_n(z) + \frac{1}{4\pi} \sum_{p \in \mathcal{P}} \int_{-\infty}^{\infty} f_p(r) \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) dr,$$
$$g(z) = \sum_{n=0}^{\infty} g_n \varphi_n(z) + \frac{1}{4\pi} \sum_{p \in \mathcal{P}} \int_{-\infty}^{\infty} g_p(r) \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) dr.$$

Then, we have the relation

$$\int_X f(z)\overline{g(z)}\, \mu_{\mathrm{hyp}}(z) = \sum_{n=0}^{\infty} f_n \overline{g}_n + \frac{1}{4\pi} \sum_{p \in \mathcal{P}} \int_{-\infty}^{\infty} f_p(r)\overline{g_p(r)} dr.$$

Proof. Let $p, q \in \mathcal{P}$ be two parabolic fixed points. Then, from Proposition 7.1 in [8], we have

$$\frac{1}{4\pi} \left\langle \int_{-\infty}^{\infty} f_p(r) \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) dr, \int_{-\infty}^{\infty} g_q(s) \mathcal{E}_{\mathrm{par},q}(z, 1/2 + is) ds \right\rangle = \\ \delta_{p,q} \int_{-\infty}^{\infty} f_p(s)\overline{g_q(s)} ds.$$

The proof of the proposition follows directly from the above equation. □

1.7 Heat Kernels

Definition 1.7.1. For $t \in \mathbb{R}_{>0}$ and $z, w \in \mathbb{H}$, the heat kernel $K_{\mathbb{H}}(t; z, w)$ on $\mathbb{R}_{>0} \times \mathbb{H} \times \mathbb{H}$ is given by the formula

$$K_{\mathbb{H}}(t; z, w) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{d_{\mathbb{H}}(z,w)}^{\infty} \frac{re^{-r^2/4t}}{\sqrt{\cosh(r) - \cosh(d_{\mathbb{H}}(z, w))}} dr, \tag{1.10}$$

where $d_{\mathbb{H}}(z, w)$ is the hyperbolic distance between z and w.

Remark 1.7.2. From the above formula, it is easy to see that $K_{\mathbb{H}}(t; z, w)$ depends only on the hyperbolic distance $d_{\mathbb{H}}(z, w)$ between z and w. So we will denote $K_{\mathbb{H}}(t; z, w)$ by $K_{\mathbb{H}}(t; \rho)$, where $\rho = d_{\mathbb{H}}(z, w)$.
For $d_{\mathbb{H}}(z, w) = 0$, the above formula simplifies to

$$K_{\mathbb{H}}(t; z, z) = K_{\mathbb{H}}(t; 0) = \frac{1}{2\pi} \int_0^{\infty} e^{-(r^2+1/4)t} r \tanh(\pi r) dr.$$

Definition 1.7.3. For $t \in \mathbb{R}_{>0}$ and $z, w \in X$, the hyperbolic heat kernel $K_{\mathrm{hyp}}(t; z, w)$ on $\mathbb{R}_{>0} \times X \times X$ is defined as

$$K_{\mathrm{hyp}}(t; z, w) = \sum_{\gamma \in \Gamma} K_{\mathbb{H}}(t; z, \gamma w). \tag{1.11}$$

For $z, w \in X$, the hyperbolic heat kernel $K_{\mathrm{hyp}}(t; z, w)$ satisfies the differential equation

$$\left(\Delta_{\mathrm{hyp},z} + \frac{\partial}{\partial t}\right) K_{\mathrm{hyp}}(t; z, w) = 0, \tag{1.12}$$

where $\Delta_{\mathrm{hyp},z}$ denotes the hyperbolic Laplacian Δ_{hyp} acting on the variable z. Furthermore for a fixed $w \in X$, and any smooth function f on X, the hyperbolic heat kernel $K_{\mathrm{hyp}}(t; z, w)$ satisfies the equation

$$\lim_{t \to 0} \int_X K_{\mathrm{hyp}}(t; z, w) f(z)\, \mu_{\mathrm{hyp}}(z) = f(w). \tag{1.13}$$

From equations (1.12) and (1.13), it can be deduced that for a fixed $w \in X$, and for all $t > 0$, the equation holds true

$$\int_X K_{\mathrm{hyp}}(t; z, w)\, \mu_{\mathrm{hyp}}(z) = 1. \tag{1.14}$$

To simplify notation, we write $K_{\mathrm{hyp}}(t; z)$ instead of $K_{\mathrm{hyp}}(t; z, z)$, when $z = w$.

The hyperbolic heat kernel $K_{\mathrm{hyp}}(t; z, w)$ admits the spectral expansion

$$K_{\mathrm{hyp}}(t; z, w) = \sum_{n=0}^{\infty} \varphi_n(z) \varphi_n(w) e^{-\lambda_n t} +$$
$$\frac{1}{4\pi} \sum_{p \in \mathcal{P}} \int_{-\infty}^{\infty} \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) \mathcal{E}_{\mathrm{par},p}(w, 1/2 - ir) e^{-(r^2+1/4)t} dr, \tag{1.15}$$

where λ_n denotes the eigenvalue of the normalized eigenfunction $\varphi_n(z)$ and $(r^2 + 1/4)$ is the eigenvalue of the eigenfunction $\mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir)$ (see also Theorem 1.6.3).

The heat kernel $K_{\mathrm{hyp}}(t; z, w)$ satisfies the long-time and short-time asymptotics.

$$K_{\mathrm{hyp}}(t; z, w) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} = O\big(e^{-c_1 t}\big) \qquad (z,\ w\ \in X;\ t \to \infty), \tag{1.16}$$

$$K_{\mathrm{hyp}}(t; z, w) = O\big(e^{-c_2/t}\big) \qquad (z,\ w\ \in\ X;\ z \neq w;\ t \to 0), \tag{1.17}$$

$$K_{\mathrm{hyp}}(t; z) - m_z K_{\mathbb{H}}(t; 0) = O\big(e^{-c_3/t}\big) \qquad (z\ \in\ X;\ t \to 0). \tag{1.18}$$

Here, c_1, c_2, and c_3 are positive constants, which depend only on the Riemann surface X.

1.8 Free-space Green's function

Definition 1.8.1. For $z, w \in \mathbb{H}$ with $z \neq w$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, the free-space Green's function $g_{\mathbb{H},s}(z, w)$ is defined as

$$g_{\mathbb{H},s}(z, w) = g_{\mathbb{H},s}(u(z, w)) = \frac{\Gamma(s)^2}{\Gamma(2s)} u^{-s} F(s, s; 2s, -1/u),$$

where $u = u(z, w) = |z - w|^2/(4\,\mathrm{Im}(z)\,\mathrm{Im}(w))$ and $F(s, s; 2s, -1/u)$ is the hypergeometric function.

For $z, w \in \mathbb{H}$ with $z \neq w$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, the free-space Green's function $g_{\mathbb{H},s}(z, w)$ converges absolutely and uniformly.

For $z, w \in \mathbb{H}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, the free-space Green's function $g_{\mathbb{H},s}(z, w)$ admits a log-singularity along the diagonal, i.e.,

$$\lim_{w \to z} \big(g_{\mathbb{H},s}(z, w) + \log|z - w|^2\big) = O_{s,z}(1).$$

Remark 1.8.2. There is a sign error in the formula defining the free-space Green's function given by equation (1.46) in [8], i.e., the last argument $-1/u$ in the hypergeometric function has been incorrectly stated as $1/u$, which we have corrected in our definition. We have also normalized the free-space Green's function defined in [8] by multiplying it by 4π.

For $z, w \in \mathbb{H}$ with $z \neq w$ and $s = 1$, we put

$$g_{\mathbb{H}}(z, w) = g_{\mathbb{H},1}(z, w),$$

and by substituting $s = 1$ in the definition of $g_{\mathbb{H},s}(z, w)$, we get

$$g_{\mathbb{H}}(z, w) = -\log\left|\frac{z - w}{z - \overline{w}}\right|^2. \tag{1.19}$$

1.9 Automorphic Green's function

Definition 1.9.1. For $z, w \in X$ with $z \neq w$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the automorphic Green's function $g_{\mathrm{hyp},s}(z,w)$ is defined as

$$g_{\mathrm{hyp},s}(z,w) = \sum_{\gamma \in \Gamma} g_{\mathbb{H},s}(z, \gamma w).$$

The following theorem summarizes the basic properties of the automorphic Green's function.

Theorem 1.9.2. *The automorphic Green's function $g_{\mathrm{hyp},s}(z,w)$ satisfies the following properties:*

(1) For $z, w \in X$ with $z \neq w$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(s(s-1)) > 1$, we have

$$g_{\mathrm{hyp},s}(z,w) = 4\pi \int_0^\infty K_{\mathrm{hyp}}(t; z, w) e^{-s(s-1)t} dt. \tag{1.20}$$

(2) For $z, w \in X$ and $z \neq w$, the automorphic Green's function satisfies the differential equation

$$\big(\Delta_{\mathrm{hyp},z} + s(s-1)\big) g_{\mathrm{hyp},s}(z,w) = 0, \tag{1.21}$$

where $\Delta_{\mathrm{hyp},z}$ denotes the hyperbolic Laplacian Δ_{hyp} acting on the variable z, as before.

(3) For $z, w \in X$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have

$$\lim_{w \to z} \big(g_{\mathrm{hyp},s}(z,w) + \log |\vartheta_z(w)|^2 \big) = O_{s,z}(1),$$

i.e, for $z \in X \backslash \mathcal{T}$, $w \in X$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have

$$\lim_{w \to z} \big(g_{\mathrm{hyp},s}(z,w) + \log |z - w|^2 \big) = O_{s,z}(1),$$

and for $z = t \in \mathcal{T}$, $w \in X$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have

$$\lim_{w \to t} \big(g_{\mathrm{hyp},s}(t,w) + \log |t - w|^{2m_t} \big) = O_{s,t}(1).$$

(4) For a fixed $w \in X$, as $z \in X$ approaches $p \in \mathcal{P}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we obtain

$$\lim_{z \to p} g_{\mathrm{hyp},s}(z,w) = 0.$$

(5) For $z, w \in X$ and $z \neq w$, the automorphic Green's function $g_{\mathrm{hyp},s}(z,w)$ admits a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = 1$ with residue $4\pi / \mathrm{vol}_{\mathrm{hyp}}(X)$, and the Laurent expansion at $s = 1$ is of the form

$$g_{\mathrm{hyp},s}(z,w) = \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X) s} \cdot \frac{1}{s-1} + g^{(1)}_{\mathrm{hyp}}(z,w) + O_{z,w}(s-1), \tag{1.22}$$

where $g^{(1)}_{\mathrm{hyp}}(z,w)$ is the constant term of $g_{\mathrm{hyp},s}(z,w)$ at $s = 1$.

Proof. The above statements are well-known, and one can find the proofs in chapters 5 and 6 of [8]. $\square$

Definition 1.9.3. Let $p, q \in \mathcal{P}$ be two parabolic fixed points. Put

$$C_{p,q} = \min\left\{ c > 0 \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_p^{-1}\Gamma\sigma_q \right\}$$

and $C_p = C_{p,p}$, where σ_p, σ_q are given by equation (1.1) in Section 1.1.

The following theorem, which gives the Fourier expansion of the automorphic Green's function, is later used for computing the Fourier expansion, and studying the behavior of the hyperbolic Green's function at the parabolic fixed points. It is also used in Chapters 4 and 7 to study the behavior of certain automorphic functions at the parabolic fixed points.

Theorem 1.9.4. *Let $p, q \in \mathcal{P}$ be two parabolic fixed points. Then for $z, w \in X$ with $\mathrm{Im}(w) > \mathrm{Im}(z)$ and $\mathrm{Im}(w)\,\mathrm{Im}(z) > C_{p,q}^{-2}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the automorphic Green's function admits the Fourier expansion*

$$\begin{aligned} g_{\mathrm{hyp},s}(\sigma_p z, \sigma_q w) &= \frac{4\pi\,\mathrm{Im}(w)^{1-s}}{2s-1}\big(\delta_{p,q}\,\mathrm{Im}(z)^s + \alpha_{p,q}(s)\,\mathrm{Im}(z)^{1-s}\big) + \\ &\frac{4\pi\,\mathrm{Im}(z)^{1-s}}{2s-1}\sum_{m\neq 0}\alpha_{p,q}(m,s)W_s(mw) + \frac{4\pi\,\mathrm{Im}(w)^{1-s}}{2s-1}\sum_{n\neq 0}\alpha_{p,q}(n,s)\overline{W_{\overline{s}}(nz)} + \\ &\delta_{p,q}\sum_{n\neq 0}\frac{1}{|n|}W_s(nw)\overline{V_{\overline{s}}(nz)} + 4\pi\sum_{mn\neq 0}Z_s(m,n)W_s(mw)\overline{W_{\overline{s}}(nz)}, \end{aligned} \tag{1.23}$$

where $\alpha_{p,q}(s)$, $\alpha_{p,q}(n,s)$, and $W_s(z)$ are as in Theorem 1.5.5; $V_s(z)$ is the Whittaker function given by equation (A.5), and $Z_s(m,n)$ is given by equation (5.16) in [8].

Proof. We refer the reader to Theorem 5.3 in [8] for the proof. $\square$

An estimate of the automorphic Green's function $g_{\mathrm{hyp},s}(\sigma_p z, \sigma_q w)$ was derived in Lemma 5.4 in [8] using Theorem 1.9.4. But one of the expressions on the right-hand side of equation (1.23) was wrongly estimated. In the following corollary, we correct this estimate.

Corollary 1.9.5. *With hypotheses as in Theorem 1.9.4, the Fourier expansion of $g_{\mathrm{hyp},s}(\sigma_p z, \sigma_q w)$ given by equation (1.23) can be further simplified to*

$$\begin{aligned} g_{\mathrm{hyp},s}(\sigma_p z, \sigma_q w) = 4\pi\frac{\mathrm{Im}(w)^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},q}(\sigma_p z, s) - \delta_{p,q}\log\big|1 - e^{2\pi i(w-z)}\big|^2 + \\ O\big(e^{-2\pi(\mathrm{Im}(w)-\mathrm{Im}(z))}\big). \end{aligned} \tag{1.24}$$

Proof. In the proof of Lemma 5.4 in [8], excepting the term

$$\delta_{p,q}\sum_{n\neq 0}\frac{1}{|n|}W_s(nw)\overline{V_{\overline{s}}(nz)}$$

appearing in the third line on the right-hand side of equation (1.23), all other terms have been correctly estimated. Considering the estimates derived in the proof of Lemma 5.4 from [8] for the remaining terms on the right-hand side of equation (1.23), we arrive at the estimate of the automorphic Green's function

$$g_{\mathrm{hyp},s}(\sigma_p z, \sigma_q w) = 4\pi \frac{\mathrm{Im}(w)^{1-s}}{2s-1} \mathcal{E}_{\mathrm{par},q}(\sigma_p z, s) + \sum_{n\neq 0} \frac{1}{|n|} W_s(nw) \overline{V_{\overline{s}}(nz)} + O\bigl(e^{-2\pi \,\mathrm{Im}(\sigma_p^{-1} w)}\bigr). \tag{1.25}$$

So to prove the lemma, it is sufficient to show that

$$\sum_{n\neq 0} \frac{1}{|n|} W_s(nw) \overline{V_{\overline{s}}(nz)} = -\log\bigl|1 - e^{2\pi i (w-z)}\bigr|^2 + O\bigl(e^{-2\pi (\mathrm{Im}(w) - \mathrm{Im}(z))}\bigr).$$

We apply the asymptotics (see proof of Lemma 5.4 in [8] for details)

$$W_s(nw) = e^{(2\pi i n \,\mathrm{Re}(w) - 2\pi |n| \,\mathrm{Im}(w))} \cdot \bigl(1 + O(|n|^{-1})\bigr),$$
$$\overline{V_{\overline{s}}(nz)} = e^{(-2\pi i n \,\mathrm{Re}(z) + 2\pi |n| \,\mathrm{Im}(z))} \cdot \bigl(1 + O(|n|^{-1})\bigr),$$

and arrive at

$$\sum_{n\neq 0} \frac{1}{|n|} W_s(nw) \overline{V_{\overline{s}}(nz)} = \sum_{n\neq 0} \frac{1}{|n|} e^{(2\pi i n \,\mathrm{Re}(w) - 2\pi |n| \,\mathrm{Im}(w))} \times e^{(-2\pi i n \,\mathrm{Re}(z) + 2\pi |n| \,\mathrm{Im}(z))} + O\bigl(e^{-2\pi (\mathrm{Im}(w) - \mathrm{Im}(z))}\bigr).$$

So it suffices to prove that

$$\sum_{n\neq 0} \frac{1}{|n|} e^{(2\pi i n \,\mathrm{Re}(w) - 2\pi |n| \,\mathrm{Im}(w))} \cdot e^{(-2\pi i n \,\mathrm{Re}(z) + 2\pi |n| \,\mathrm{Im}(z))} = -\log\bigl|1 - e^{2\pi i (w-z)}\bigr|^2.$$

The left-hand side in the above equation can be written as a sum of two expressions

$$\sum_{n\neq 0} \frac{1}{|n|} e^{(2\pi i n \,\mathrm{Re}(w) - 2\pi |n| \,\mathrm{Im}(w))} \cdot e^{(-2\pi i n \,\mathrm{Re}(z) + 2\pi |n| \,\mathrm{Im}(z))} = \sum_{n>0} \frac{1}{n} e^{2\pi i n (\mathrm{Re}(w) + i \,\mathrm{Im}(w))} \cdot e^{2\pi i n (-\mathrm{Re}(z) - i \,\mathrm{Im}(z))} + \sum_{n<0} \frac{-1}{n} e^{2\pi i n (\mathrm{Re}(w) - i \,\mathrm{Im}(w))} \cdot e^{2\pi i n (-\mathrm{Re}(z) + i \,\mathrm{Im}(z))}. \tag{1.26}$$

The first expression on the right-hand side of equation (1.26) can be written as

$$\sum_{n>0} \frac{1}{n} e^{2\pi i n (\mathrm{Re}(w) + i \,\mathrm{Im}(w))} \cdot e^{2\pi i n (-\mathrm{Re}(z) - i \,\mathrm{Im}(z))} = \sum_{n>0} \frac{1}{n} e^{2\pi i n (\mathrm{Re}(w) + i \,\mathrm{Im}(w) - \mathrm{Re}(z) - i \,\mathrm{Im}(z))} = \sum_{n>0} \frac{1}{n} e^{2\pi i n (w-z)}. \tag{1.27}$$

Since

$$\left|e^{2\pi in(w-z)}\right| = \left|e^{2\pi in(\mathrm{Re}(w)-\mathrm{Re}(z))}\right| \cdot e^{-2\pi n(\mathrm{Im}(w)-\mathrm{Im}(z))} < 1,$$

from the Taylor expansion of $-\log|1-z|$, we get

$$\sum_{n>0} \frac{1}{n} e^{2\pi in(w-z)} = -\log\big(1 - e^{2\pi i(w-z)}\big).$$

Similarly, after replacing the variable n by $-m$, the second expression in equation (1.26) simplifies to

$$\begin{aligned}
&\sum_{n<0} \frac{-1}{n} e^{2\pi in(\mathrm{Re}(w)-i\,\mathrm{Im}(w))} \cdot e^{2\pi in(-\mathrm{Re}(z)+i\,\mathrm{Im}(z))} = \\
&\sum_{m>0} \frac{1}{m} e^{-2\pi im(\mathrm{Re}(w)-\mathrm{Re}(z)-i\,\mathrm{Im}(w)+i\,\mathrm{Im}(z))} = \\
&\sum_{m>0} \frac{1}{m} e^{-2\pi im(\overline{w}-\overline{z})} = -\log\big(1 - e^{-2\pi i(\overline{w}-\overline{z})}\big).
\end{aligned} \tag{1.28}$$

Hence, combining equations (1.27) and (1.28), we get

$$\begin{aligned}
&\sum_{n\neq 0} \frac{1}{|n|} e^{(2\pi in\,\mathrm{Re}(w)-2\pi|n|\,\mathrm{Im}(w))} \cdot e^{(-2\pi in\,\mathrm{Re}(z)+2\pi|n|\,\mathrm{Im}(z))} = \\
&-\log\big(1 - e^{2\pi i(w-z)}\big) - \log\big(1 - e^{-2\pi i(\overline{w}-\overline{z})}\big) = -\log\big|1 - e^{2\pi i(w-z)}\big|^2,
\end{aligned}$$

which proves the lemma. □

1.10 Hyperbolic Green's function

Definition 1.10.1. For $z, w \in X$ and $z \neq w$, the hyperbolic Green's function is defined as

$$g_{\mathrm{hyp}}(z,w) = 4\pi \int_0^\infty \left(K_{\mathrm{hyp}}(t;z,w) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right) dt.$$

The following theorem states the basic properties of the hyperbolic Green's function, which directly follow from its definition.

Theorem 1.10.2. *The hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ satisfies the following properties:*

(1) For $z, w \in X$ and $z \neq w$, $g_{\mathrm{hyp}}(z,w)$ is smooth and symmetric in z and w.

(2) For $z, w \in X$, we have

$$\lim_{w\to z} \big(g_{\mathrm{hyp}}(z,w) + \log|\vartheta_z(w)|^2\big) = O_z(1),$$

for $z \in X \backslash \mathcal{T}$ and $w \in X$, we have

$$\lim_{w\to z} \big(g_{\mathrm{hyp}}(z,w) + \log|z-w|^2\big) = O_z(1),$$

and for $z = t \in \mathcal{T}$ and $w \in X$, we have

$$\lim_{w \to t} \left(g_{\mathrm{hyp}}(z,w) + \log|z-t|^{2m_t} \right) = O_t(1).$$

(3) For $z, w \in X \backslash \mathcal{T}$, the hyperbolic Green's function satisfies the differential equation

$$d_z d_z^c g_{\mathrm{hyp}}(z,w) + \delta_w(z) = \mu_{\mathrm{shyp}}(z), \tag{1.29}$$

with the normalization condition

$$\int_X g_{\mathrm{hyp}}(z,w)\, \mu_{\mathrm{hyp}}(z) = 0. \tag{1.30}$$

(4) For $z, w \in X$ and $z \neq w$, we have

$$g_{\mathrm{hyp}}^{(1)}(z,w) = g_{\mathrm{hyp}}(z,w). \tag{1.31}$$

Proof. The first three properties are well-known, and one can easily deduce them either from the properties of the heat kernel mentioned in Section 1.7, or from the properties of the automorphic Green's function $g_{\mathrm{hyp},s}(z,w)$ discussed in Section 1.9.

From equations (1.20) and (1.22), we find that

$$\begin{aligned} g_{\mathrm{hyp}}^{(1)}(z,w) &= \lim_{s \to 1} \left(g_{\mathrm{hyp},s}(z,w) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)s} \cdot \frac{1}{s-1} \right) \\ &= 4\pi \lim_{s \to 1} \left(\int_0^\infty K_{\mathrm{hyp}}(t;z,w) e^{-s(s-1)} dt - \int_0^\infty \frac{e^{-s(s-1)}}{\mathrm{vol}_{\mathrm{hyp}}(X)} dt \right) \\ &= 4\pi \int_0^\infty \left(K_{\mathrm{hyp}}(t;z,w) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) dt \\ &= g_{\mathrm{hyp}}(z,w), \end{aligned}$$

which proves assertion (4) and hence, the theorem. □

1.11 Key identity

The following proposition will be useful in Section 2.7 for computing the first Chern form with respect to the residual hyperbolic metric on $\Omega_{\overline{X}}^1$.

Proposition 1.11.1. *For $z \in X \backslash \mathcal{T}$, we have*

$$\begin{aligned} &- d_z d_z^c \lim_{w \to z} \left(g_{\mathrm{hyp}}(z,w) + \log|z-w|^2 \right) = \\ &\frac{1}{2\pi} \mu_{\mathrm{hyp}}(z) + \left(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z) dt \right) \mu_{\mathrm{hyp}}(z). \end{aligned}$$

Proof. When X admits no torsion and no parabolic fixed points, the result has been proved as Proposition 3.3 in [11]. The same proof can be easily adapted to our case, provided $z \in X \backslash \mathcal{T}$. $\square$

The following theorem gives a very important identity which relates the canonical and hyperbolic metrics. Using computations of Chern forms, the result has been proved as Theorem 3.4 in [11], for the case when X admits no torsion and no parabolic fixed points.

In [10], using Theorem 3.4 in [11] and studying the degeneration of compact Riemann surfaces, the result has been extended to the case when X admits parabolic fixed points.

Theorem 1.11.2. *For $z \in X \backslash \mathcal{T}$, we have the relation of differential forms*

$$g\,\mu_{\mathrm{can}}(z) = \\ \left(\frac{1}{4\pi} + \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\mu_{\mathrm{hyp}}(z) + \frac{1}{2}\left(\int_0^\infty \Delta_{\mathrm{hyp}}\, K_{\mathrm{hyp}}(t;z)dt\right)\mu_{\mathrm{hyp}}(z).$$

Proof. As stated above, the result has been established as Theorem 3.4 in [11], when X is compact. The proof given in [11] applies to our case where X does admit torsion and parabolic fixed points, as long as $z \in X \backslash \mathcal{T}$. $\square$

Chapter 2

Key identity for currents

As mentioned before, Theorem 1.11.2 which relates the canonical and hyperbolic metrics, has been first proved in [11] as Theorem 3.4. In [11], in Lemma 3.1, J. Jorgenson and J. Kramer first expressed the difference of the canonical and hyperbolic Green's functions, in terms of integrals involving only the hyperbolic Green's function and the canonical metric.

Then in Proposition 3.3, they compute the first Chern form with respect to the residual hyperbolic metric on the cotangent bundle. Then using the computation of the first Chern form with respect to the residual canonical metric on the cotangent bundle from [2], they establish Theorem 3.4.

In [10], J. Jorgenson and J. Kramer even extended Theorem 1.11.2 to non-compact, finite volume Riemann surfaces without torsion points. They proved the extension by studying Theorem 3.4 from [11] for a degenerating family of compact hyperbolic Riemann surfaces.

In this chapter, we extend the relation of differential forms in Theorem 1.11.2 to torsion and parabolic fixed points at the level of currents. We follow the original method of proof given in [11], as the computations carried out in this article still remain valid away from the torsion and parabolic fixed points. We also use computations and results from [14].

In Section 2.1 we describe the extensions of the hyperbolic and canonical metrics to $\overline{X}$.

In Section 2.2, we introduce the canonical Green's function on $\overline{X}\times\overline{X}$, and then show that its restriction to $X \times X$ is the canonical Green's function studied in Section 1.4.

We also state a distributional relation from [17], which the current associated to the canonical Green's function satisfies on $\overline{X}$.

In Section 2.3, we introduce the residual canonical metric, and state the first Chern form with respect to this metric on the cotangent bundle of $\overline{X}$.

In Section 2.4, using the Fourier expansion of the automorphic Green's function described in Section 1.9, we derive the asymptotics of the hyperbolic Green's function at the parabolic fixed points. These asymptotics are very useful in the analysis that follows.

In Section 2.5, using the analysis of Section 2.4, we describe an extension of the hyperbolic Green's function to $\overline{X}$. We then show that the current $[\widehat{g}_{\mathrm{hyp}}(\cdot, w)]$ associated to the hyperbolic Green's function $g_{\mathrm{hyp}}(z, w)$ defines a Green's current on $\overline{X}$, for every $w \in \overline{X} \backslash \mathcal{P}$.

In Section 2.6, we prove an auxiliary identity, which expresses the difference of the canonical and hyperbolic Green's functions, in terms of integrals involving only the hyperbolic Green's function and the canonical metric.

In Section 2.7, we introduce the residual hyperbolic metric, and compute the first Chern form with respect to this metric on the cotangent bundle of $\overline{X}$.

In Section 2.8, we recall results from [14], where it has been shown that the function

$$\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t; z) dt$$

remains bounded on $\overline{X}$.

In Section 2.9, using the results from previous sections, we derive an extension of Theorem 1.11.2 to torsion and parabolic fixed points at the level of currents.

2.1 Extensions of μ_{hyp} and μ_{can} to $\overline{X}$

Remark 2.1.1. From equations (1.3) and (1.4), we find that the hyperbolic (1,1)-form $\mu_{\mathrm{hyp}}(z)$ becomes singular, but still remains integrable on $\overline{X}$. We denote this singular and integrable (1,1)-form on $\overline{X}$ by $\widehat{\mu}_{\mathrm{hyp}}(z)$.

Definition 2.1.2. Since the hyperbolic (1,1)-form $\widehat{\mu}_{\mathrm{hyp}}(z)$ is integrable on $\overline{X}$, it defines a current $[\widehat{\mu}_{\mathrm{hyp}}(z)]$ of type $(1,1)$ on $\overline{X}$. The current $[\widehat{\mu}_{\mathrm{hyp}}(z)]$ acts on smooth functions f defined on $\overline{X}$ in the usual way, i.e.,

$$[\widehat{\mu}_{\mathrm{hyp}}(z)](f) = \int_{\overline{X}} f(z)\, \widehat{\mu}_{\mathrm{hyp}}(z).$$

Since $\widehat{\mu}_{\mathrm{hyp}}(z)$ is integrable at the parabolic fixed points, and since there are only finitely many of them, the volume of $\overline{X}$ with respect to the extended hyperbolic volume form $\widehat{\mu}_{\mathrm{hyp}}(z)$, is the same as that of X with respect to $\mu_{\mathrm{hyp}}(z)$. So we denote it again by $\mathrm{vol}_{\mathrm{hyp}}(X)$.

The rescaled hyperbolic (1,1)-form is given by

$$\widehat{\mu}_{\mathrm{shyp}}(z) = \frac{\widehat{\mu}_{\mathrm{hyp}}(z)}{\mathrm{vol}_{\mathrm{hyp}}(X)},$$

which measures the volume of $\overline{X}$ to be one. Let $[\widehat{\mu}_{\mathrm{shyp}}(z)]$ denote the current defined by the rescaled hyperbolic (1,1)-form $\widehat{\mu}_{\mathrm{shyp}}(z)$ on $\overline{X}$.

Remark 2.1.3. Let $\widehat{\mu}_{\mathrm{can}}(z)$ denote (1,1)-form corresponding to the canonical metric on $\overline{X}$. Then the canonical (1,1)-form $\mu_{\mathrm{can}}(z)$ is the restriction of the (1,1)-form $\widehat{\mu}_{\mathrm{can}}(z)$ to X.

Definition 2.1.4. The canonical (1,1)-form $\widehat{\mu}_{\mathrm{can}}(z)$ defines a current $[\widehat{\mu}_{\mathrm{can}}(z)]$ of type $(1,1)$ on $\overline{X}$, acting on smooth functions f defined on $\overline{X}$ in the usual way, i.e.,

$$[\widehat{\mu}_{\mathrm{can}}(z)](f) = \int_{\overline{X}} f(z)\, \widehat{\mu}_{\mathrm{can}}(z).$$

2.2 Extension of $g_{\mathrm{can}}(z,w)$ to $\overline{X}$

Definition 2.2.1. For $z,w \in \overline{X}$, the canonical Green's function $\widehat{g}_{\mathrm{can}}(z,w)$ is defined as the solution of the differential equation

$$d_z d_z^c \widehat{g}_{\mathrm{can}}(z,w) + \delta_w(z) = \widehat{\mu}_{\mathrm{can}}(z), \tag{2.1}$$

with the normalization condition

$$\int_{\overline{X}} \widehat{g}_{\mathrm{can}}(z,w)\, \widehat{\mu}_{\mathrm{can}}(z) = 0. \tag{2.2}$$

The canonical Green's function $\widehat{g}_{\mathrm{can}}(z,w)$ admits a log-singularity along the diagonal, i.e., for $z,w \in \overline{X}$, we have

$$\lim_{w\to z} \left(\widehat{g}_{\mathrm{can}}(z,w) + \log|\vartheta_z(w)|^2 \right) = O_z(1). \tag{2.3}$$

In [2], Arakelov has proved the existence, uniqueness, and symmetry of the canonical Green's function $\widehat{g}_{\mathrm{can}}(z,w)$, for all compact Riemann surfaces.

Lemma 2.2.2. *The canonical Green's function $g_{\mathrm{can}}(z,w)$ is the restriction of $\widehat{g}_{\mathrm{can}}(z,w)$ to $X \times\ X$.*

Proof. It is easy to see that for $z,w \in \overline{X}\backslash\mathcal{P}$, the canonical Green's function $\widehat{g}_{\mathrm{can}}(z,w)$ satisfies the differential equation (1.6). The canonical metric $\widehat{\mu}_{\mathrm{can}}(z)$ remains smooth at the parabolic fixed points, so for $w \in \overline{X}\backslash\mathcal{P}$ bounded away from the parabolic fixed points, $\widehat{g}_{\mathrm{can}}(z,w)$ remains smooth, as z approaches the parabolic fixed points.

Since $\overline{X}$ is obtained by adding finitely many parabolic fixed points to X, we can conclude that $\widehat{g}_{\mathrm{can}}(z,w)$ also satisfies the normalization condition, i.e., equation (1.7), which implies that $g_{\mathrm{can}}(z,w)$ is the restriction of $\widehat{g}_{\mathrm{can}}(z,w)$ to $X\times X$. $\square$

Corollary 2.2.3. *The canonical Green's function $g_{\mathrm{can}}(z,w)$ exists, and is unique. Furthermore, for $z,w \in X$ and $z \neq w$, $g_{\mathrm{can}}(z,w)$ is symmetric in z and w.*

Proof. The proof follows from Lemma 2.2.2, and from the existence, uniqueness, and symmetry of $\widehat{g}_{\mathrm{can}}(z,w)$. $\square$

Definition 2.2.4. For a fixed $w \in \overline{X}$, the canonical Green's function $\widehat{g}_{\mathrm{can}}(z,w)$ defines a function on $\overline{X}$ with log-singularity at $z = w$, and remains smooth for all z bounded away from w. So it is integrable with respect to smooth (1,1)-forms η defined on $\overline{X}$, and hence, defines a current $[\widehat{g}_{\mathrm{can}}(\cdot,w)]$ of type $(0,0)$ on $\overline{X}$. Its action on smooth (1,1)-forms η is given by

$$[\widehat{g}_{\mathrm{can}}(\cdot,w)](\eta) = \int_{\overline{X}} \widehat{g}_{\mathrm{can}}(z,w)\eta(z).$$

The following lemma shows that the current $[\widehat{g}_{\mathrm{can}}(\cdot,w)]$ associated to the canonical Green's function $\widehat{g}_{\mathrm{can}}(z,w)$ is a Green's current, for a fixed $w \in \overline{X}$.

Lemma 2.2.5. *For a fixed $w \in \overline{X}$, we have the distributional relation on $\overline{X}$*

$$d_z d_z^c [\widehat{g}_{\mathrm{can}}(z,w)] + \delta_w(z) = [\widehat{\mu}_{\mathrm{can}}(z)].$$

Proof. This result follows from Theorem II.1.5 in [17]. □

2.3 Residual canonical metric on $\Omega^1_{\overline{X}}$

Definition 2.3.1. For $z \in \overline{X}$, we define

$$\log \|d\vartheta_z\|^2_{\mathrm{res,can}}(z) = \lim_{w \to z} \big(\widehat{g}_{\mathrm{can}}(z,w) + \log|\vartheta_z(w)|^2\big).$$

Since the function

$$\lim_{w \to z} \big(\widehat{g}_{\mathrm{can}}(z,w) + \log|\vartheta_z(w)|^2\big)$$

remains smooth for all $z \in \overline{X}$, the residual canonical metric is well defined and smooth on $\overline{X}$.

Definition 2.3.2. Since $\log \|d\vartheta_z\|^2_{\mathrm{res,can}}(z)$ remains smooth on $\overline{X}$, it defines a current $\big[\log \|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big]$ of type (0,0) on $\overline{X}$.

The following proposition gives the first Chern form associated to the residual canonical metric on the cotangent bundle $\Omega^1_{\overline{X}}$.

Proposition 2.3.3. *For $z \in \overline{X}$, the first Chern form $c_1\big(\Omega^1_{\overline{X}}, \|\cdot\|_{\mathrm{res,can}}\big)$ is given by the formula*

$$c_1\big(\Omega^1_{\overline{X}}, \|\cdot\|_{\mathrm{res,can}}\big) = -d_z d_z^c \log \|d\vartheta_z\|^2_{\mathrm{res,can}}(z) = (2g-2)\,\widehat{\mu}_{\mathrm{can}}(z).$$

Proof. We refer the reader to [2] for the details of the proof. □

2.4 $g_{\text{hyp}}(z, w)$ at the parabolic fixed points

In the following proposition, using the Fourier expansion of the automorphic Green's function stated in Proposition 1.9.4, we compute the Fourier expansion of the hyperbolic Green's function.

Using the Fourier expansion of the hyperbolic Green's function, we ascertain its behavior at the parabolic fixed points in the corollaries that follow.

Proposition 2.4.1. *For a fixed $w \in X$, and for $z \in X$ with $\operatorname{Im}(\sigma_p^{-1}z) > \operatorname{Im}(\sigma_p^{-1}w)$ and $\operatorname{Im}(\sigma_p^{-1}z)\operatorname{Im}(\sigma_p^{-1}w) > C_p^{-2}$, we have*

$$g_{\text{hyp}}(z, w) = 4\pi\kappa_p(w) - \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)} - \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)} \log\big(\operatorname{Im}(\sigma_p^{-1}z)\big) - \log\big|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\big|^2 + O\big(e^{-2\pi(\operatorname{Im}(\sigma_p^{-1}z) - \operatorname{Im}(\sigma_p^{-1}w))}\big), \tag{2.4}$$

where σ_p is a scaling matrix associated to the parabolic fixed point $p \in \mathcal{P}$ given by equation (1.1), and C_p is as defined in Section 1.9.

Proof. From the proof of Theorem 1.10.2, we have

$$\begin{aligned} g_{\text{hyp}}(z, w) &= \lim_{s\to 1}\left(g_{\text{hyp},s}(w, z) - \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)s} \cdot \frac{1}{s-1}\right) \\ &= \lim_{s\to 1}\left(g_{\text{hyp},s}(w, z) - \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)} \cdot \frac{1}{s-1}\right) + \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)}. \end{aligned} \tag{2.5}$$

Now for a fixed $w \in X$, for each $z \in X$ with $\operatorname{Im}(\sigma_p^{-1}z) > \operatorname{Im}(\sigma_p^{-1}w)$ and $\operatorname{Im}(\sigma_p^{-1}z)\operatorname{Im}(\sigma_p^{-1}w) > C_p^{-2}$, and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, from Corollary 1.9.5, we have

$$g_{\text{hyp},s}(z, w) = 4\pi\frac{\operatorname{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1}\mathcal{E}_{\text{par},p}(w, s) - \log\big|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\big|^2 + O\big(e^{-2\pi(\operatorname{Im}(\sigma_p^{-1}z) - \operatorname{Im}(\sigma_p^{-1}w))}\big). \tag{2.6}$$

Since the limit in equation (2.5) converges uniformly, we can substitute the automorphic Green's function $g_{\text{hyp,s}}(z, w)$ in equation (2.5) by the expression on the right-hand side of equation (2.6), and we get

$$g_{\text{hyp}}(z, w) = 4\pi\lim_{s\to 1}\left(\frac{\operatorname{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1}\mathcal{E}_{\text{par},p}(w, s) - \frac{1}{\operatorname{vol}_{\text{hyp}}(X)} \cdot \frac{1}{s-1}\right) + \frac{4\pi}{\operatorname{vol}_{\text{hyp}}(X)} - \log\big|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\big|^2 + O\big(e^{-2\pi(\operatorname{Im}(\sigma_p^{-1}z) - \operatorname{Im}(\sigma_p^{-1}w))}\big). \tag{2.7}$$

To evaluate the limit

$$4\pi\lim_{s\to 1}\left(\frac{\operatorname{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1}\mathcal{E}_{\text{par},p}(w, s) - \frac{1}{\operatorname{vol}_{\text{hyp}}(X)} \cdot \frac{1}{s-1}\right),$$

we need to compute the Laurent expansions of $\mathcal{E}_{\mathrm{par},p}(w,s)$, $\mathrm{Im}(\sigma_p^{-1}z)^{1-s}$, and $(2s-1)^{-1}$ at $s=1$. The Laurent expansions of $\mathrm{Im}\,(\sigma_p^{-1}z)^{1-s}$ and $(2s-1)^{-1}$ at $s=1$ are easy to compute, and are of the form

$$\begin{aligned}
&\mathrm{Im}\,(\sigma_p^{-1}z)^{1-s} = 1-(s-1)\log\big(\mathrm{Im}\,(\sigma_p^{-1}z)\big)+O\big((s-1)^2\big),\\
&\frac{1}{2s-1} = 1-2(s-1)+O\big((s-1)^2\big);
\end{aligned}$$

from Theorem 1.5.2, we know that the Laurent expansion of $\mathcal{E}_{\mathrm{par},p}(w,s)$ at $s=1$ is of the form

$$\mathcal{E}_{\mathrm{par},p}(w,s)=\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\cdot\frac{1}{s-1}+\kappa_p(w)+O_w(s-1).$$

Combining the above three equations, we arrive at

$$\begin{aligned}
&4\pi\lim_{s\to1}\left(\frac{\mathrm{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},p}(w,s)-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\cdot\frac{1}{s-1}\right)=\\
&4\pi\kappa_p(w)-\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big),
\end{aligned}\tag{2.8}$$

which together with equation (2.7) implies the proposition. □

Corollary 2.4.2. *For a fixed $w\in X$, as $z\in X$ approaches a parabolic fixed point $p\in\mathcal{P}$, we have*

$$\begin{aligned}
g_{\mathrm{hyp}}(z,w)&=-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)+O_{z,w}(1)\\
&=-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(-\log|\vartheta_p(z)|\big)+O_{z,w}(1),
\end{aligned}\tag{2.9}$$

where the contribution from the term $O_{z,w}(1)$ in the above equation is a smooth function in z.

Proof. The corollary follows from Proposition 2.4.1. □

Corollary 2.4.3. *For $p,q\in\mathcal{P}$ and $p\neq q$, and $z,w\in X$ approaching p,q, respectively, we have*

$$\begin{aligned}
&g_{\mathrm{hyp}}(z,w)=\\
&-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_q^{-1}w)\big)+O_{z,w}(1)=\\
&-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(-\log|\vartheta_p(z)|\big)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(-\log|\vartheta_q(w)|\big)+O_{z,w}(1),
\end{aligned}$$

where the contribution from the term $O_{z,w}(1)$ in the above equation is a smooth function in z and w.

Proof. We first let $z \in X$ approach $p \in \mathcal{P}$, and then allow $w \in X$ approach $q \in \mathcal{P}$. From Proposition 2.4.1, we know that for a fixed $w \in X$, as $z \in X$ approaches $p \in \mathcal{P}$, we get

$$g_{\mathrm{hyp}}(z,w) = 4\pi\kappa_p(w) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\big(\mathrm{Im}(\sigma_p^{-1}z)\big) + O_{z,w}(1). \tag{2.10}$$

From Corollary 1.5.4, as $w \in X$ approaches $q \in \mathcal{P}$ with $q \neq p$, we find

$$\kappa_p(w) = -\frac{\log\big(\mathrm{Im}({\sigma_q}^{-1}w)\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)} + O_w(1). \tag{2.11}$$

Now the corollary follows by just combining equations (2.10) and (2.11). $\square$

2.5 Extension of $g_{\mathrm{hyp}}(z,w)$ to $\overline{X}$

In this section, we define the current associated to the hyperbolic Green's function, and then proceed to prove that it is a Green's current.

Definition 2.5.1. From Corollary 2.4.2, we know that for a fixed $w \in X$, the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ has a log log-growth as z approaches a parabolic fixed point. Hence, for a fixed $w \in \overline{X}\backslash\mathcal{P}$, it defines a singular function $\widehat{g}_{\mathrm{hyp}}(z,w)$ on $\overline{X}$ with a log log-singularity at the parabolic fixed points and a log-singularity at $z = w$.

Definition 2.5.2. For a fixed $w \in \overline{X}\backslash\mathcal{P}$, we know that $\widehat{g}_{\mathrm{hyp}}(z,w)$ admits a log log-singularity at the parabolic fixed points and a log-singularity at $z = w$. So it is integrable with respect to smooth (1,1)-forms defined on $\overline{X}$, and hence, defines a current $[\widehat{g}_{\mathrm{hyp}}(\cdot,w)]$ of type (0,0) on $\overline{X}$. It acts on smooth (1,1)-forms η defined on $\overline{X}$ in the usual way, i.e.,

$$[\widehat{g}_{\mathrm{hyp}}(\cdot,w)](\eta) = \int_{\overline{X}} \widehat{g}_{\mathrm{hyp}}(z,w)\eta(z).$$

Analogously, for a fixed $z \in \overline{X}\backslash\mathcal{P}$, the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ extends to a singular function $\widehat{g}_{\mathrm{hyp}}(z,w)$ on $\overline{X}$, and defines a current $[\widehat{g}_{\mathrm{hyp}}(z,\cdot)]$ of type (0,0) on $\overline{X}$.

Since $g_{\mathrm{hyp}}(z,w)$ is symmetric in z and w, we expect the currents $[\widehat{g}_{\mathrm{hyp}}(\cdot,w)]$ and $[\widehat{g}_{\mathrm{hyp}}(w,\cdot)]$ to be equal, which is indeed the case as shown in the following lemma.

Lemma 2.5.3. *For a fixed $w \in \overline{X}\backslash\mathcal{P}$, we have the relation of currents*

$$[\widehat{g}_{\mathrm{hyp}}(w,\cdot)] = [\widehat{g}_{\mathrm{hyp}}(\cdot,w)].$$

Proof. For any smooth (1,1)-form η on $\overline{X}$, we need to prove that

$$\int_{\overline{X}} \widehat{g}_{\mathrm{hyp}}(w,z)\eta(z) = \int_{\overline{X}} \widehat{g}_{\mathrm{hyp}}(z,w)\eta(z).$$

Let $U_r(w)$, $U_r(p)$ denote open coordinate disks of radius r around w, a parabolic fixed point $p \in \mathcal{P}$, respectively. Put

$$Y_r = \overline{X} \backslash \Big(U_r(w) \cup \bigcup_{p\in\mathcal{P}} U_r(p) \Big).$$

For z, $w \in X$ and $z \neq w$, $g_{\mathrm{hyp}}(z,w)$ is symmetric in z and w. So we get

$$\int_{Y_r} \widehat{g}_{\mathrm{hyp}}(w,z)\eta(z) = \int_{Y_r} \widehat{g}_{\mathrm{hyp}}(z,w)\eta(z).$$

We now choose r small enough such that for $z \in U_r(w)$, we have

$$\widehat{g}_{\mathrm{hyp}}(w,z) = -\log|z-w|^{2m_w} + O_{r,w}(1) = \widehat{g}_{\mathrm{hyp}}(z,w),$$

and for $z \in U_r(p)$ and $p \in \mathcal{P}$, from Corollary 2.4.2, we have

$$\widehat{g}_{\mathrm{hyp}}(w,z) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\big(\mathrm{Im}(\sigma_p^{-1}z)\big) + O_{r,w}(1) = \widehat{g}_{\mathrm{hyp}}(z,w).$$

Hence, we arrive at

$$\int_{U_r(w)} \widehat{g}_{\mathrm{hyp}}(w,z)\eta(z) = \int_{U_r(w)} \widehat{g}_{\mathrm{hyp}}(z,w)\eta(z),$$

and

$$\sum_{p\in\mathcal{P}} \int_{U_r(p)} \widehat{g}_{\mathrm{hyp}}(w,z)\eta(z) = \sum_{p\in\mathcal{P}} \int_{U_r((p)} \widehat{g}_{\mathrm{hyp}}(z,w)\eta(z),$$

which finishes the proof. $\square$

In analogy to Lemma 2.2.5, we derive the following relation of currents on $\overline{X}$. In the following lemma, we show that the current $[\widehat{g}_{\mathrm{hyp}}(\cdot,w)]$, associated to the hyperbolic Green's function $\widehat{g}_{\mathrm{hyp}}(z,w)$ is a Green's current.

Lemma 2.5.4. *For a fixed $w \in \overline{X}\backslash\mathcal{P}$, we have the relation of currents on $\overline{X}$*

$$d_z d_z^c[\widehat{g}_{\mathrm{hyp}}(z,w)] + \delta_w(z) = [\widehat{\mu}_{\mathrm{shyp}}(z)].$$

Proof. We need to show that for a fixed $w \in \overline{X}\backslash\mathcal{P}$, and any smooth function f on $\overline{X}$, the equality holds true

$$\int_{\overline{X}} \widehat{g}_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) + f(w) = \int_{\overline{X}} f(z)\, \widehat{\mu}_{\mathrm{shyp}}(z).$$

We will mimic the proof of Theorem II.1.5 in [17]. We divide the proof into two cases.

Case 1. Let $w \in \overline{X} \backslash \mathcal{S}$ and $U_r(w)$, $U_r(t)$, $U_r(p)$ denote open coordinate disks of radius r around w, a torsion point $t \in \mathcal{T}$, and a parabolic fixed point $p \in \mathcal{P}$, respectively. Put

$$Y_r = \overline{X} \backslash \biggl(U_r(w) \cup \bigcup_{t \in \mathcal{T}} U_r(t) \cup \bigcup_{p \in \mathcal{P}} U_r(p) \biggr).$$

So we find that

$$\begin{aligned}
&\int_{Y_r} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) - \int_{Y_r} f(z)\, \mu_{\mathrm{shyp}}(z) \xrightarrow[r \to 0]{} \\
&\int_{\overline{X}} \widehat{g}_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) - \int_{\overline{X}} f(z)\, \widehat{\mu}_{\mathrm{shyp}}(z) = \\
&d_z d_z^c [\widehat{g}_{\mathrm{hyp}}(z,w)](f) - [\widehat{\mu}_{\mathrm{shyp}}(z)](f).
\end{aligned}$$

From equation (1.29), we have

$$\begin{aligned}
&\int_{Y_r} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) - \int_{Y_r} f(z)\, \mu_{\mathrm{shyp}}(z) = \\
&\int_{Y_r} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) - \int_{Y_r} f(z) d_z d_z^c g_{\mathrm{hyp}}(z,w).
\end{aligned}$$

So it suffices to prove that

$$\int_{Y_r} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) - \int_{Y_r} f(z) d_z d_z^c g_{\mathrm{hyp}}(z,w) \xrightarrow[r \to 0]{} -f(w). \tag{2.12}$$

From Stokes's theorem, we have

$$\begin{aligned}
&\int_{Y_r} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) - \int_{Y_r} f(z) d_z d_z^c g_{\mathrm{hyp}}(z,w) = \\
&\int_{\partial U_r(w)} f(z) d_z^c g_{\mathrm{hyp}}(z,w) - \int_{\partial U_r(w)} g_{\mathrm{hyp}}(z,w) d_z^c f(z) + \\
&\sum_{t \in \mathcal{T}} \int_{\partial U_r(t)} f(z) d_z^c g_{\mathrm{hyp}}(z,w) - \sum_{t \in \mathcal{T}} \int_{\partial U_r(t)} g_{\mathrm{hyp}}(z,w) d_z^c f(z) + \\
&\sum_{p \in \mathcal{P}} \int_{\partial U_r(p)} f(z) d_z^c g_{\mathrm{hyp}}(z,w) - \sum_{p \in \mathcal{P}} \int_{\partial U_r(p)} g_{\mathrm{hyp}}(z,w) d_z^c f(z).
\end{aligned}$$

Now for $z \in U_r(w)$, we use d_z^c in polar coordinates, which is given by

$$d_z^c = \frac{r}{2} \frac{\partial}{\partial r} \frac{d\theta}{2\pi} - \frac{1}{4\pi} \frac{\partial}{\partial \theta} \frac{dr}{r}. \tag{2.13}$$

From Theorem 1.10.2, we know that for $z \in \partial U_r(w)$

$$g_{\mathrm{hyp}}(z,w) = -\log r^2 + O_{r,w}(1),$$

where the contribution from the term $O_{r,w}(1)$ in the above equation is a smooth function in r. So we derive

$$\int_{\partial U_r(w)} f(z)d^c_z g_{\mathrm{hyp}}(z,w) - \int_{\partial U_r(w)} g_{\mathrm{hyp}}(z,w)d^c_z f(z) =$$
$$-\int_{\partial U_r(w)} f(z)r\frac{\partial \log r}{\partial r}\frac{d\theta}{2\pi} + \int_{\partial U_r(w)} r\log r\frac{\partial f}{\partial r}\frac{d\theta}{2\pi} + O(r).$$

Since f is a smooth function on $\overline{X}$, and

$$r\frac{\partial \log r}{\partial r} \xrightarrow[r\to 0]{} 1 \quad\text{and}\quad r\log r \xrightarrow[r\to 0]{} 0,$$

we get

$$\int_{\partial U_r(w)} f(z)d^c_z g_{\mathrm{hyp}}(z,w) - \int_{\partial U_r(w)} g_{\mathrm{hyp}}(z,w)d^c_z f(z) \xrightarrow[r\to 0]{} -f(w). \tag{2.14}$$

For $z\in U_r(t)$ and $w\in \overline{X}\backslash\mathcal{S}$, the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ remains smooth. Since f is a smooth function on $\overline{X}$, we get

$$\int_{\partial U_r(t)} f(z)d^c_z g_{\mathrm{hyp}}(z,w) - \int_{\partial U_r(t)} g_{\mathrm{hyp}}(z,w)d^c_z f(z) \xrightarrow[r\to 0]{} 0. \tag{2.15}$$

From Corollary 2.4.2, we know that for $z\in\partial U_r(p)$

$$g_{\mathrm{hyp}}(z,w) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(-\log r\big) + O_{r,w}(1),$$

where the contribution from the term $O_{r,w}(1)$ in the above equation is a smooth function in r. So using polar coordinates stated in equation (2.13), we derive

$$\int_{\partial U_r(p)} f(z)d^c_z g_{\mathrm{hyp}}(z,w) - \int_{\partial U_r(p)} g_{\mathrm{hyp}}(z,w)d^c_z f(z) =$$
$$-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\int_{\partial U_r(p)} f(z)\frac{r}{2}\frac{\partial\log\big(-\log r\big)}{\partial r}\frac{d\theta}{2\pi} +$$
$$\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\int_{\partial U_r(p)} \frac{r}{2}\log\big(-\log r\big)\frac{\partial f}{\partial r}\frac{d\theta}{2\pi} + O(r).$$

Again since f is smooth on $\overline{X}$, and

$$r\frac{\partial\log\big(-\log r\big)}{\partial r} = \frac{1}{\log r} \xrightarrow[r\to 0]{} 0 \quad\text{and}\quad r\log\big(-\log r\big) \xrightarrow[r\to 0]{} 0, \tag{2.16}$$

we get

$$\int_{\partial U_r(p)} f(z)d^c_z g_{\mathrm{hyp}}(z,w) - \int_{\partial U_r(p)} g_{\mathrm{hyp}}(z,w)d^c_z f(z) \xrightarrow[r\to 0]{} 0. \tag{2.17}$$

The above equation also holds true for $w=t\in\mathcal{T}$.

Combining equations (2.14), (2.15), and (2.17), we arrive at equation (2.12), which proves the lemma for the case $w \in \overline{X}\backslash\mathcal{S}$.

Case 2. Let $w = u \in \mathcal{T}$. Since equation (2.15) remains valid for $w = u \in \mathcal{T}$, it suffices to prove that

$$\sum_{t\in\mathcal{T}}\int_{\partial U_r(t)} f(z)d_z^c g_{\mathrm{hyp}}(z,u) - \sum_{t\in\mathcal{T}}\int_{\partial U_r(t)} g_{\mathrm{hyp}}(z,u)d_z^c f(z) \xrightarrow[r\to 0]{} -f(u).$$

For $z \in \partial U_r(t)$ and $t \neq u$, $g_{\mathrm{hyp}}(z,u)$ remains smooth. Since f is a smooth function on $\overline{X}$, we get

$$\int_{\partial U_r(t)} f(z)d_z^c g_{\mathrm{hyp}}(z,u) - \int_{\partial U_r(t)} g_{\mathrm{hyp}}(z,u)d_z^c f(z) \xrightarrow[r\to 0]{} 0. \tag{2.18}$$

For $z \in \partial U_r(u)$, from Theorem 1.10.2, we can express $g_{\mathrm{hyp}}(z,u)$ as

$$g_{\mathrm{hyp}}(z,u) = -\log|z-u|^{2m_u} + O_{r,u}(1),$$

which in local coordinates around u transforms to

$$g_{\mathrm{hyp}}(z,u) = -\log|\vartheta_u(z)|^2 + O_{r,u}(1) = -\log r^2 + O_{r,u}(1),$$

where again the contribution from the term $O_{r,u}(1)$ in the above equation is a smooth function in r. This implies that Case 2 is analogous to Case 1, and hence, the proof follows. □

Definition 2.5.5. Using the fact that $\widehat{g}_{\mathrm{hyp}}(z,w)$ admits a log log-singularity at the parabolic fixed points, it can be shown that the limit $\lim_{w\to p}[\widehat{g}_{\mathrm{hyp}}(\cdot,w)]$ does not exist, where $p \in \mathcal{P}$. But from Lemma 2.5.4, we find that $d_z d_z^c[\widehat{g}_{\mathrm{hyp}}(z,w)]$ still exists. So for $w = p \in \mathcal{P}$, we put

$$d_z d_z^c[\widehat{g}_{\mathrm{hyp}}(z,p)] = [\widehat{\mu}_{\mathrm{shyp}}(z)] - \delta_p(z).$$

Proposition 2.5.6. *For a fixed $w \in \overline{X}$, we have the relation of currents on $\overline{X}$*

$$d_z d_z^c[\widehat{g}_{\mathrm{hyp}}(z,w)] + \delta_w(z) = [\widehat{\mu}_{\mathrm{shyp}}(z)].$$

Proof. The proof follows from Lemma 2.5.4 and Definition 2.5.5. □

2.6 An auxiliary identity

When X is compact, and devoid of any torsion and parabolic fixed points, Lemma 3.1 in [11] expresses the difference of the hyperbolic and canonical Green's functions as a sum of integrals involving the hyperbolic Green's function and the canonical metric. In this section, we extend this lemma to the case when X admits torsion and parabolic fixed points.

Lemma 2.6.1. *There exists a unique integrable function $\widehat{\Phi}(z)$ defined on $\overline{X}$ satisfying the equation*

$$d_z d_z^c[\widehat{\Phi}(z)] = [\widehat{\mu}_{\mathrm{shyp}}(z)] - [\widehat{\mu}_{\mathrm{can}}(z)], \tag{2.19}$$

with the normalization condition

$$\int_X \widehat{\Phi}(z)\,\mu_{\mathrm{can}}(z) = 0, \tag{2.20}$$

where $[\widehat{\Phi}(z)]$ is the current determined by $\widehat{\Phi}(z)$ on $\overline{X}$.

Proof. Since the cohomology classes of $[\widehat{\mu}_{\mathrm{shyp}}(z)]$ and $[\widehat{\mu}_{\mathrm{can}}(z)]$ are equal in $H^2\left(\overline{X},\mathbb{Z}\right) \cong \mathbb{Z}$, the difference

$$[\widehat{\mu}_{\mathrm{shyp}}(z)] - [\widehat{\mu}_{\mathrm{can}}(z)]$$

is a d-exact current on $\overline{X}$. Hence, from the $\partial\bar{\partial}$-lemma for currents, we can conclude that there exists an integrable function $\widehat{\Phi}(z)$ defined on $\overline{X}$ such that

$$d_z d_z^c[\widehat{\Phi}(z)] = [\widehat{\mu}_{\mathrm{shyp}}(z)] - [\widehat{\mu}_{\mathrm{can}}(z)],$$

which proves the existence of $\widehat{\Phi}(z)$.

Let $\widehat{\Psi}(z)$ be another integrable function on $\overline{X}$ satisfying equations (2.19) and (2.20). Then

$$d_z d_z^c[\widehat{\Phi}(z) - \widehat{\Psi}(z)] = 0,$$

which implies that $[\widehat{\Phi}(z) - \widehat{\Psi}(z)]$ is a harmonic current on $\overline{X}$, and since $\overline{X}$ is compact, we get

$$\widehat{\Phi}(z) = \widehat{\Psi}(z) + C,$$

for some constant function C on $\overline{X}$. Since both $\widehat{\Phi}(z)$ and $\widehat{\Psi}(z)$ satisfy equation (2.20), it implies that the constant $C = 0$, and hence the function $\widehat{\Phi}(z)$ is uniquely determined by equations (2.19) and (2.20). $\square$

Lemma 2.6.2. *Let us denote the restriction of $\widehat{\Phi}(z)$ to X by $\Phi(z)$. Then, for $z, w \in X$, we have*

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \\ \frac{1}{2}\bigg(\Phi(z) + \Phi(w) + \int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) + \int_X g_{\mathrm{hyp}}(w,\zeta)\,\mu_{\mathrm{can}}(\zeta)\bigg). \tag{2.21}$$

Proof. For a fixed $w \in X$, consider the function

$$F_w(z) = g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) - \int_X g_{\mathrm{hyp}}(w,\zeta)\,\mu_{\mathrm{can}}(\zeta)$$

defined on X.

For a fixed $w \in X$, we know that $g_{\mathrm{hyp}}(z,w)$ is integrable with respect to smooth (1,1)-forms, and determines a current of type (0,0) on $\overline{X}$. Moreover, the canonical metric $\widehat{\mu}_{\mathrm{can}}(z)$ remains smooth at the parabolic fixed points, and since there are only finitely many parabolic fixed points, we can conclude that

$$\int_{\overline{X}} \widehat{g}_{\mathrm{hyp}}(w,\zeta)\, \widehat{\mu}_{\mathrm{can}}(\zeta) = \int_X g_{\mathrm{hyp}}(w,\zeta)\, \mu_{\mathrm{can}}(\zeta),$$

and hence, the integral

$$\int_X g_{\mathrm{hyp}}(w,\zeta)\, \mu_{\mathrm{can}}(\zeta)$$

exists. Furthermore,

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)$$

remains bounded as z approaches w. So we can conclude that the function $F_w(z)$ is well defined on X.

For a fixed $w \in X$, we have seen in Sections 2.2 and 2.5 that the Green's functions $g_{\mathrm{can}}(z,w)$ and $g_{\mathrm{hyp}}(z,w)$ extend to $\overline{X}$ and define currents $[\widehat{g}_{\mathrm{can}}(\cdot,w)]$ and $[\widehat{g}_{\mathrm{hyp}}(\cdot,w)]$, respectively, of type (0,0). Hence, for a fixed $w \in X$, the function $F_w(z)$ determines a current

$$[\widehat{F}_w] = [\widehat{g}_{\mathrm{hyp}}(\cdot,w)] - [\widehat{g}_{\mathrm{can}}(\cdot,w)] - \int_X g_{\mathrm{hyp}}(w,\zeta)\, \mu_{\mathrm{can}}(\zeta)$$

of type (0,0) on $\overline{X}$. It follows from Lemmas 2.2.5 and 2.5.4 that $[\widehat{F}_w]$ satisfies equation (2.19).

It is also easy to see that $F_w(z)$ satisfies equation (2.20). Hence, from the uniqueness of $\widehat{\Phi}(z)$, we get

$$\Phi(z) = F_w(z) = g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) - \int_X g_{\mathrm{hyp}}(w,\zeta)\, \mu_{\mathrm{can}}(\zeta), \tag{2.22}$$

which implies that $F_w(z)$ is independent of $w \in X$.

From the symmetry of the Green's functions $g_{\mathrm{hyp}}(z,w)$ and $g_{\mathrm{can}}(z,w)$, it follows that

$$\Phi(w) = g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) - \int_X g_{\mathrm{hyp}}(z,\zeta)\, \mu_{\mathrm{can}}(\zeta), \tag{2.23}$$

and combining equations (2.22) and (2.23), we arrive at

$$\begin{aligned} &g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \\ &\frac{1}{2}\bigg(\Phi(z) + \Phi(w) + \int_X g_{\mathrm{hyp}}(z,\zeta)\, \mu_{\mathrm{can}}(\zeta) + \int_X g_{\mathrm{hyp}}(w,\zeta)\, \mu_{\mathrm{can}}(\zeta)\bigg), \end{aligned}$$

which proves the lemma. □

Remark 2.6.3. From equation (2.22), we have

$$\int_X g_{\mathrm{hyp}}(\xi,\zeta)\, \mu_{\mathrm{can}}(\zeta) = g_{\mathrm{hyp}}(z,\xi) - g_{\mathrm{can}}(z,\xi) - \Phi(z),$$

for $\xi, z \in X$. Since the integral of each term on the right-hand side of the above equation exists when integrated with respect to $\mu_{\mathrm{can}}(\xi)$ on X, we can conclude that the integral

$$\int_X \int_X g_{\mathrm{hyp}}(\xi,\zeta)\,\mu_{\mathrm{can}}(\zeta)\,\mu_{\mathrm{can}}(\xi)$$

exists.

Proposition 2.6.4. *For $z, w \in X$, we have*

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \phi(z) + \phi(w),$$

where

$$\phi(z) = \int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) - \frac{1}{2}\int_X \int_X g_{\mathrm{hyp}}(\xi,\zeta)\,\mu_{\mathrm{can}}(\zeta)\,\mu_{\mathrm{can}}(\xi).$$

Proof. For all $z, w \in X$, from Lemma 2.6.2, we have

$$2\int_X \big(g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)\big)\,\mu_{\mathrm{can}}(w) =$$
$$\int_X \left(\Phi(z) + \Phi(w) + \int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) + \int_X g_{\mathrm{hyp}}(w,\zeta)\,\mu_{\mathrm{can}}(\zeta)\right)\mu_{\mathrm{can}}(w),$$

which due to equation (2.20), further simplifies to

$$2\int_X g_{\mathrm{hyp}}(z,w)\,\mu_{\mathrm{can}}(w) =$$
$$\Phi(z) + \int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) + \int_X \int_X g_{\mathrm{hyp}}(\xi,\zeta)\,\mu_{\mathrm{can}}(\zeta)\,\mu_{\mathrm{can}}(\xi).$$

Hence, we arrive at

$$\int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) = \Phi(z) + \int_X \int_X g_{\mathrm{hyp}}(\xi,\zeta)\,\mu_{\mathrm{can}}(\zeta)\,\mu_{\mathrm{can}}(\xi),$$

which leads to

$$\Phi(z) = \int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) - \int_X \int_X g_{\mathrm{hyp}}(\xi,\zeta)\,\mu_{\mathrm{can}}(\zeta)\,\mu_{\mathrm{can}}(\xi).$$

Substituting the above formula for $\Phi(z)$ in equation (2.21), we get

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) +$$
$$\int_X g_{\mathrm{hyp}}(w,\zeta)\,\mu_{\mathrm{can}}(\zeta) - \int_X \int_X g_{\mathrm{hyp}}(\xi,\zeta)\,\mu_{\mathrm{can}}(\zeta)\,\mu_{\mathrm{can}}(\xi).$$

The proof of the proposition follows by setting

$$\phi(z) = \int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) - \frac{1}{2}\int_X \int_X g_{\mathrm{hyp}}(\xi,\zeta)\,\mu_{\mathrm{can}}(\zeta)\,\mu_{\mathrm{can}}(\xi).$$

□

Using Corollary 2.4.2, we derive the behavior of the function $\phi(z)$ at the parabolic fixed points in the following corollary.

Corollary 2.6.5. *As $z \in X$ approaches a parabolic fixed point $p \in \mathcal{P}$, we have*

$$\phi(z) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\big(-\log|\vartheta_p(z)|\big) + O_z(1), \tag{2.24}$$

where the contribution from the term $O_z(1)$ in the above equation is a smooth function in z.

Proof. From Proposition 2.6.4, we know that for $z \in X$,

$$\phi(z) = g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) - \phi(w),$$

for any $w \in X$, fixed. The proof now follows from Corollary 2.4.2, and the fact that the canonical Green's function $g_{\mathrm{can}}(z,w)$ remains smooth, as z approaches a parabolic fixed point $p \in \mathcal{P}$. □

We can give an alternate proof for Corollary 2.4.3, using Proposition 2.6.4 and Corollary 2.6.5.

Corollary 2.6.6. *For $p, q \in \mathcal{P}$ and $p \neq q$, and $z, w \in X$ approaching p, q, respectively, we have*

$$g_{\mathrm{hyp}}(z,w) = \\ -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\big(-\log|\vartheta_p(z)|\big) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\big(-\log|\vartheta_q(w)|\big) + O_{z,w}(1),$$

where the contribution from the term $O_{z,w}(1)$ in the above equation is a smooth function in z and w.

Proof. From Proposition 2.6.4, we have

$$g_{\mathrm{hyp}}(z,w) = g_{\mathrm{can}}(z,w) + \phi(z) + \phi(w).$$

Now the proof easily follows from Corollary 2.6.5, and the fact that the canonical Green's function $g_{\mathrm{can}}(z,w)$ remains bounded, as z, w approach p, q, respectively, for $p \neq q$. □

Remark 2.6.7. From Corollary 2.6.5, we know that the function $\phi(z)$ is smooth on X, with a $\log\log$-growth at the parabolic fixed points. So $\phi(z) \in L^2(X)$, i.e., the function $\phi(z)$ is square integrable with respect to the hyperbolic metric $\mu_{\mathrm{hyp}}(z)$ on X. Hence, from Theorem 1.6.3 it follows that $\phi(z)$ admits a spectral expansion of the form

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n \varphi_n(z) + \frac{1}{4\pi} \sum_{p \in \mathcal{P}} \int_{-\infty}^{\infty} \phi_p(r) \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) dr,$$

where $\phi_n = \big\langle \phi(z), \varphi_n(z) \big\rangle$ and $\phi_p(r) = \big\langle \phi(z), \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) \big\rangle$, and the inner product $\langle \cdot, \cdot \rangle$ is as defined in Section 1.6.

2.7 Residual hyperbolic metric on $\Omega^1_{\overline{X}}$

In this section, we introduce the residual hyperbolic metric on the cotangent bundle $\Omega^1_{\overline{X}}$. We show that it is $\log\log$-singular at the parabolic fixed points, and then compute the first Chern form associated to the residual hyperbolic metric.

Definition 2.7.1. For $z \in X$, define

$$\log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) = \lim_{w\to z}\big(g_{\mathrm{hyp}}(z,w) + \log|\vartheta_z(w)|^2\big).$$

Remark 2.7.2. From Proposition 1.10.2, we know that

$$\lim_{w\to z}\big(g_{\mathrm{hyp}}(z,w) + \log|\vartheta_z(w)|^2\big)$$

remains bounded for all $z \in X$, so we can conclude that the residual hyperbolic metric is well defined on X.

In the following proposition, we show that the residual hyperbolic metric is $\log\log$-singular at the parabolic fixed points.

Proposition 2.7.3. *As $z \in X$ approaches a parabolic fixed point $p \in \mathcal{P}$, we have*

$$\log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) = -\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(-\log|\vartheta_p(z)|\big) + O_z(1),$$

where the contribution from the term $O_z(1)$ in the above equation is a smooth function in z.

Proof. Let $z \in U_r(p)$, where $U_r(p)$ denotes an open disk of radius r around the parabolic fixed point $p \in \mathcal{P}$. For a fixed $w \in X$ and r small enough, from Proposition 2.4.1, we have

$$\begin{aligned} g_{\mathrm{hyp}}(z,w) = 4\pi\kappa_p(w) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big) - \\ \log\big|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\big|^2 + O\big(e^{-2\pi(\mathrm{Im}(\sigma_p^{-1}z) - \mathrm{Im}(\sigma_p^{-1}w))}\big). \end{aligned}$$

This implies that as w approaches z, we get

$$\begin{aligned} \lim_{w\to z}\left(g_{\mathrm{hyp}}(z,w) + \log\big|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\big|^2\right) = \\ 4\pi\kappa_p(z) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big) + O_z(1). \end{aligned}$$

Recalling the Fourier expansion of the Kronecker's limit function $\kappa_p(z)$ described in Corollary 1.5.4, and combining it with the above equation, we get

$$\begin{aligned} \lim_{w\to z}\left(g_{\mathrm{hyp}}(z,w) + \log\big|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\big|^2\right) = \\ 4\pi\left(\mathrm{Im}(\sigma_p^{-1}z) - \frac{2}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)\right) + O_z(1), \end{aligned} \tag{2.25}$$

where the contribution from the term $O_z(1)$ in the above equation is a smooth function in z. Now for $z \in U_r(p)$, we have

$$\log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) = \lim_{w\to z} \big(g_{\mathrm{hyp}}(z,w) + \log|\vartheta_z(w)|^2\big) =$$
$$\lim_{w\to z} \left(g_{\mathrm{hyp}}(z,w) + \log\left|e^{2\pi i\sigma_p^{-1}w} - e^{2\pi i\sigma_p^{-1}z}\right|^2\right) =$$
$$\lim_{w\to z} \left(g_{\mathrm{hyp}}(z,w) + \log\left|1 - e^{2\pi i(\sigma_p^{-1}z-\sigma_p^{-1}w)}\right|^2 + \log\left|e^{2\pi i\sigma_p^{-1}w}\right|^2\right) =$$
$$\lim_{w\to z} \left(g_{\mathrm{hyp}}(z,w) + \log\left|1 - e^{2\pi i(\sigma_p^{-1}z-\sigma_p^{-1}w)}\right|^2\right) - 4\pi\,\mathrm{Im}(\sigma_p^{-1}z), \tag{2.26}$$

and combining equations (2.25) and (2.26), we arrive at

$$\log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) = -\frac{8\pi}{\mathrm{vol_{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big) + O_z(1) =$$
$$-\frac{8\pi}{\mathrm{vol_{hyp}}(X)}\log\big(-\log|\vartheta_p(z)|\big) + O_z(1),$$

which completes the proof of the proposition. □

Remark 2.7.4. One can also prove the above proposition using Proposition 2.6.4 and Corollary 2.6.5. From Proposition 2.6.4, we have

$$\lim_{w\to z}\big(g_{\mathrm{hyp}}(z,w) + \log|\vartheta_z(w)|^2\big) = \lim_{w\to z}\big(g_{\mathrm{can}}(z,w) + \log|\vartheta_z(w)|^2\big) + 2\phi(z).$$

Since

$$\lim_{w\to z}\big(\widehat{g}_{\mathrm{can}}(z,w) + \log|\vartheta_z(w)|^2\big)$$

remains bounded, as z approaches a parabolic fixed point $p \in \mathcal{P}$, Proposition 2.7.3 follows directly from Corollary 2.6.5.

Definition 2.7.5. From Proposition 2.7.3, it follows that $\log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)$ is smooth on $\overline{X}\backslash\mathcal{P}$, and admits a $\log\log$-singularity at the parabolic fixed points. So it remains integrable at parabolic fixed points, and hence, defines a current $\big[\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)\big]$ of type (0,0) on $\overline{X}$.

Using Proposition 1.11.1, we compute the first Chern form associated to the residual hyperbolic metric in the following proposition.

Proposition 2.7.6. *For $z \in \overline{X}\backslash\mathcal{S}$, the first Chern form $c_1\big(\Omega^1_{\overline{X}}, \|\cdot\|_{\mathrm{res,hyp}}\big)$ is given by the formula*

$$c_1\big(\Omega^1_{\overline{X}}, \|\cdot\|_{\mathrm{res,hyp}}\big) = -d_z d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) =$$
$$\frac{1}{2\pi}\,\mu_{\mathrm{hyp}}(z) + \left(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;z)dt\right)\mu_{\mathrm{hyp}}(z).$$

Proof. For $z \in \overline{X}$, away from the torsion points and parabolic fixed points, we have

$$\lim_{w\to z}\big(g_{\mathrm{hyp}}(z,w) + \log|\vartheta_z(w)|^2\big) = \lim_{w\to z}\big(g_{\mathrm{hyp}}(z,w) + \log|z-w|^2\big),$$

and then the proposition follows directly from Proposition 1.11.1. □

2.8 Some convergence and boundedness results

The function

$$\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z)dt,$$

is one of the ingredients of the key identity described in Theorem 1.11.2. We recall results from [14], where this function has been shown to be bounded on X, and also at the parabolic fixed points.

The boundedness of the above mentioned quantity is crucial for the extension of the key identity to torsion and parabolic fixed points at the level of currents.

Proposition 2.8.1. *For $z \in X$, the series*

$$\sum_{\substack{\gamma\in\Gamma\backslash\{\mathrm{id}\}\\ \gamma\,\mathrm{parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)$$

is absolutely and locally uniformly convergent, and remains bounded at the parabolic fixed points.

Proof. We refer the reader to Lemma 5.2 in [14], for the proof. □

Proposition 2.8.2. *For $z \in X$, the series*

$$\sum_{\substack{\gamma\in\Gamma\backslash\{\mathrm{id}\}\\ \gamma\,\mathrm{torsion}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)$$

is absolutely and locally uniformly convergent, and remains bounded at the parabolic fixed points.

Proof. The proof follows from Lemma 6.3 in [14]. □

Proposition 2.8.3. *For $z \in X$*

$$\sum_{\substack{\gamma\in\Gamma\backslash\{\mathrm{id}\}\\ \gamma\,\mathrm{hyperbolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)$$

is absolutely and locally uniformly convergent, and remains bounded at the parabolic fixed points.

Proof. The proof follows from Proposition 7.2 in [14]. □

Theorem 2.8.4. *For all $z \in X\backslash\mathcal{T}$, we have the relation*

$$4\pi\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z)dt = \sum_{\gamma\in\Gamma\backslash\{\mathrm{id}\}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z). \tag{2.27}$$

Furthermore, the series

$$\sum_{\gamma\in\Gamma\backslash\{\mathrm{id}\}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)$$

remains bounded, as z approaches a torsion point or a parabolic fixed point.

Proof. In [14], equation (2.27) has been proved as Lemma 8.2.
From Propositions 2.8.1, 2.8.2, and 2.8.3, it follows that we can write

$$\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) = \sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) +$$

$$\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{torsion}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) + \sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{hyperbolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z),$$

and conclude that the series

$$\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)$$

remains bounded at the torsion and parabolic fixed points, which completes the proof of the theorem. □

Definition 2.8.5. From Theorem 2.8.4, we know that

$$\int_0^\infty \Delta_{\mathrm{hyp}}\, K_{\mathrm{hyp}}(t;z)dt,$$

defines a smooth function on $\overline{X}$, which we denote symbolically by

$$\int_0^\infty \Delta_{\mathrm{hyp}}\, \widehat{K}_{\mathrm{hyp}}(t;z)dt.$$

Hence, we can conclude that the differential form

$$\left(\int_0^\infty \Delta_{\mathrm{hyp}}\, \widehat{K}_{\mathrm{hyp}}(t;z)dt\right)\widehat{\mu}_{\mathrm{hyp}}(z)$$

defines a current

$$\left[\left(\int_0^\infty \Delta_{\mathrm{hyp}}\, \widehat{K}_{\mathrm{hyp}}(t;z)dt\right)\widehat{\mu}_{\mathrm{hyp}}(z)\right]$$

of type (1,1) on $\overline{X}$.

2.9 Extension of the key identity to currents

In this section, we extend Theorem 1.11.2, which relates the hyperbolic and canonical metrics to a relation of currents on $\overline{X}$.

We use the following notation only until the end of this chapter.

Notation 2.9.1. Let $U_{r_0}(s)$, denote an open coordinate disk of fixed radius r_0 around a torsion or a parabolic fixed point $s \in \mathcal{S}$, and r_0 is small enough such that any two coordinate disks are disjoint. Put

$$U_{r_0} = \bigcup_{s\in\mathcal{S}} U_{r_0}(s) \ \text{ and } \ Y_{r_0} = \overline{X}\setminus U_{r_0}.$$

Furthermore, let $U_r(s)$ denote an open coordinate disk of radius r around $s \in \mathcal{S}$ and $0 < r < r_0$. Put

$$U_r = \bigcup_{s\in\mathcal{S}} U_r(s) \;\text{ and }\; U_{r_0,r} = U_{r_0}\backslash U_r.$$

The following proposition is an extension of Proposition 2.3.3 to torsion and parabolic fixed points at the level of currents.

Proposition 2.9.2. *For f a smooth function on U_{r_0}, we have the relation of currents on U_{r_0}*

$$-d_zd_z^c\big[\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big](f) = (2g-2)[\widehat{\mu}_{\mathrm{can}}(z)](f)+$$
$$\int_{\partial U_{r_0}}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z^cf(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)).$$

Proof. To prove the proposition it suffices to prove that

$$\int_{U_{r_0,r}}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_zd_z^cf(z)) - (2g-2)\int_{U_{r_0,r}} f(z)\,\mu_{\mathrm{can}}(z) \xrightarrow[r\to 0]{}$$
$$\int_{\partial U_{r_0}}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z^cf(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)). \tag{2.28}$$

From Proposition 2.3.3, for all $z \in U_{r_0,r}$, we have by restriction

$$-d_zd_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z) = (2g-2)\,\widehat{\mu}_{\mathrm{can}}(z).$$

Using the above relation, the left-hand side of the limit considered in (2.28) simplifies to the expression

$$\int_{U_{r_0,r}}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_zd_z^cf(z)) - \int_{U_{r_0,r}} f(z)(-d_zd_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)).$$

By Stokes's theorem, the above expression further decomposes into the four terms

$$\int_{\partial U_{r_0}}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z^cf(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z))+$$
$$\int_{\partial U_r}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)d_z^cf(z) - \int_{\partial U_r} f(z)d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z).$$

Since $\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)$ and the function f are both smooth on U_{r_0}, the third and fourth terms in the above expression yield

$$\int_{\partial U_r}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)d_z^cf(z) - \int_{\partial U_r} f(z)d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z) \xrightarrow[r\to 0]{} 0.$$

This concludes the proof of the proposition. $\square$

The following proposition is an extension of Proposition 2.7.6 to torsion and parabolic fixed points at the level of currents.

Proposition 2.9.3. *For f a smooth function on U_{r_0}, we have the relation of currents on U_{r_0}*

$$
\begin{aligned}
&-d_z d_z^c\big[\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)\big](f) = \\
&\frac{1}{2\pi}[\widehat{\mu}_{\mathrm{hyp}}(z)](f) + \bigg[\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\bigg)\,\widehat{\mu}_{\mathrm{hyp}}(z)\bigg](f)+ \\
&\int_{\partial U_{r_0}}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)).
\end{aligned}
$$

Proof. To prove the proposition we need to show that

$$
\begin{aligned}
&\int_{U_{r_0,r}}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z d_z^c f(z)) - \frac{1}{2\pi}\int_{U_{r_0,r}} f(z)\,\mu_{\mathrm{hyp}}(z) - \\
&\int_{U_{r_0,r}} f(z)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;z)dt\bigg)\,\mu_{\mathrm{hyp}}(z) \xrightarrow[r\to 0]{} \\
&\int_{\partial U_{r_0}}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)).
\end{aligned}
\tag{2.29}
$$

By Proposition 2.7.6, for all $z \in U_{r_0,r}$, we have

$$
-d_z d_z^c\log\|dz\|^2_{\mathrm{res,hyp}}(z) = \frac{1}{2\pi}\,\mu_{\mathrm{hyp}}(z) + \bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;z)dt\bigg)\,\mu_{\mathrm{hyp}}(z).
$$

So the expression on the left-hand side of the limit considered in (2.29) simplifies to the expression

$$
\int_{U_{r_0,r}}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z d_z^c f(z)) - \int_{U_{r_0,r}} f(z)(-d_z d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)).
$$

By Stokes's theorem, the above expression decomposes into the four terms

$$
\begin{aligned}
&\int_{\partial U_{r_0}}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z))+ \\
&\int_{\partial U_r}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d_z^c f(z) - \int_{\partial U_r} f(z)d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z).
\end{aligned}
\tag{2.30}
$$

So it suffices to prove that as r approaches zero, the expression in the second line of (2.30) converges to zero, i.e.,

$$
\begin{aligned}
&\int_{\partial U_r}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d_z^c f(z) - \int_{\partial U_r} f(z)d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) = \\
&\sum_{s\in\mathcal{S}}\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d_z^c f(z) - \\
&\sum_{s\in\mathcal{S}}\int_{\partial U_r(s)} f(z)d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) \xrightarrow[r\to 0]{} 0.
\end{aligned}
\tag{2.31}
$$

For each $s \in \mathcal{S}$, we now consider the limit

$$\lim_{r\to 0}\bigg(\int_{\partial U_r(s)} \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) d_z^c f(z) - \int_{\partial U_r(s)} f(z) d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)\bigg).$$

Let $s = t \in \mathcal{T}$ be a torsion point. Then by Remark 2.7.2, we know that $\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)$ remains smooth on $U_{r_0}(t)$. Hence, we find that

$$\begin{aligned}&\sum_{t\in\mathcal{T}}\int_{\partial U_r(s)} \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) d_z^c f(z) - \\ &\sum_{t\in\mathcal{T}}\int_{\partial U_r(s)} f(z) d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) \xrightarrow[r\to 0]{} 0. \qquad (2.32)\end{aligned}$$

Let $s = p \in \mathcal{P}$ be a parabolic fixed point. From Proposition 2.7.3, we know that $\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)$ is log log-singular at the parabolic fixed points. For $p \in \mathcal{P}$, using the formula for d_z^c in polar coordinates described in equation (2.13), we have

$$\begin{aligned}&\int_{\partial U_r(p)} \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) d_z^c f(z) - \int_{\partial U_r(p)} f(z) d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) = \\ &-\frac{8\pi}{\mathrm{vol_{hyp}}(X)}\int_{\partial U_r(p)} \frac{r}{2}\log\big(-\log r\big)\frac{\partial f}{\partial r}\frac{d\theta}{2\pi} + \\ &\frac{8\pi}{\mathrm{vol_{hyp}}(X)}\int_{\partial U_r(p)} f(z)\frac{r}{2}\frac{\partial \log\big(-\log r\big)}{\partial r}\frac{d\theta}{2\pi} + O(r).\end{aligned}$$

From equation (2.16), and the fact that f is smooth on $U_{r_0}(p)$, we conclude that

$$\begin{aligned}&\sum_{p\in\mathcal{P}}\int_{\partial U_r(p)} \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) d_z^c f(z) - \\ &\sum_{p\in\mathcal{P}}\int_{\partial U_r(p)} f(z) d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) \xrightarrow[r\to 0]{} 0. \qquad (2.33)\end{aligned}$$

Combining (2.32) and (2.33) establishes the limit under consideration in (2.31) and hence, the proposition. □

The following proposition is an extension of Proposition 3.2 of [11] to torsion and parabolic fixed points at the level of currents.

Proposition 2.9.4. *For f a smooth function on U_{r_0}, we have the relation of currents on U_{r_0}*

$$\begin{aligned}&-d_z d_z^c\big[\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)\big](f) = 2g[\widehat{\mu}_{\mathrm{can}}(z)](f) - 2[\widehat{\mu}_{\mathrm{shyp}}(z)](f) + \\ &\int_{\partial U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)).\end{aligned} \qquad (2.34)$$

Proof. From Proposition 2.9.2, for f a smooth function on U_{r_0}, we have the relation of currents on U_{r_0}

$$
\begin{aligned}
&-d_zd_z^c\big[\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big](f)=(2g-2)[\widehat{\mu}_{\mathrm{can}}(z)](f)+\\
&\int_{\partial U_{r_0}}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z^cf(z))-\int_{\partial U_{r_0}}f(z)(-d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)).
\end{aligned}
$$

Subtracting the above equation from the desired equality in (2.34), it follows that, it is sufficient to prove that the difference of the currents

$$
(-d_zd_z^c)\big[\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)\big](f)-(-d_zd_z^c)\big[\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big](f)
$$

is equal to the expression

$$
\begin{aligned}
&2[\widehat{\mu}_{\mathrm{can}}(z)](f)-2[\widehat{\mu}_{\mathrm{shyp}}(z)](f)+\\
&\int_{\partial U_{r_0}}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z^cf(z))-\int_{\partial U_{r_0}}f(z)(-d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z))+\\
&\int_{\partial U_{r_0}}f(z)(-d_z^c\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z))-\int_{\partial U_{r_0}}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z^cf(z)).
\end{aligned}
$$

So in order to prove the proposition, we have to show that

$$
\begin{aligned}
&\int_{U_{r_0,r}}\big(\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)-\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big)(-d_zd_z^cf(z))-\\
&2\int_{U_{r_0,r}}f(z)\big(\mu_{\mathrm{can}}(z)-\mu_{\mathrm{shyp}}(z)\big)\xrightarrow[r\to0]{}\\
&\int_{\partial U_{r_0}}\big(\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)-\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big)(-d_z^cf(z))-\\
&\int_{\partial U_{r_0}}f(z)\big(-d_z^c\big(\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)-\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big)\big).
\end{aligned}\tag{2.35}
$$

For all $z,w\in U_{r_0,r}$, by Proposition 2.6.4, we have

$$
g_{\mathrm{hyp}}(z,w)-g_{\mathrm{can}}(z,w)=\phi(z)+\phi(w).
$$

For $z\in U_{r_0,r}$, taking $d_zd_z^c$ we get

$$
\mu_{\mathrm{shyp}}(z)-\mu_{\mathrm{can}}(z)=d_zd_z^c\phi(z).\tag{2.36}
$$

Furthermore, for $z\in X$, we have

$$
\begin{aligned}
&\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)-\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)=\\
&\lim_{w\to z}\big(g_{\mathrm{hyp}}(z,w)-g_{\mathrm{can}}(z,w)\big)=2\phi(z).
\end{aligned}\tag{2.37}
$$

Using equations (2.36) and (2.37), the left-hand side of (2.35) simplifies to the expression

$$\int_{U_{r_0,r}} \big(\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)-\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big)(-d_zd_z^cf(z))-$$
$$2\int_{U_{r_0,r}} f(z)\big(\mu_{\mathrm{can}}(z)-\mu_{\mathrm{shyp}}(z)\big)=$$
$$2\int_{U_{r_0,r}}\phi(z)(-d_zd_z^cf(z))-2\int_{U_{r_0,r}}f(z)(-d_zd_z^c\phi(z)).$$

By Stokes's theorem, the quantity on the right-hand side of the above equation decomposes into the four terms

$$2\int_{\partial U_{r_0}}\phi(z)(-d_z^cf(z))-2\int_{\partial U_{r_0}}f(z)(-d_z^c\phi(z))+$$
$$2\int_{\partial U_r}\phi(z)d_z^cf(z)-2\int_{\partial U_r}f(z)d_z^c\phi(z). \tag{2.38}$$

Using equation (2.37), we find that the first two terms in (2.38) give

$$\int_{\partial U_{r_0}} \big(\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)-\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big)(-d_z^cf(z))-$$
$$\int_{\partial U_{r_0}} f(z)\big(-d_z^c\big(\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)-\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big)\big).$$

So it suffices to prove that, as r approaches zero, the expression in the second line of (2.38) converges to zero, i.e.

$$2\int_{\partial U_r}\phi(z)d_z^cf(z)-2\int_{\partial U_r}f(z)d_z^c\phi(z)=$$
$$2\sum_{s\in\mathcal{S}}\left(\int_{\partial U_r(s)}\phi(z)d_z^cf(z)-\int_{\partial U_r(s)}f(z)d_z^c\phi(z)\right)\xrightarrow[r\to0]{}0. \tag{2.39}$$

For each $s\in\mathcal{S}$, we now consider the limit

$$\lim_{r\to0}\left(\int_{\partial U_r(s)}\phi(z)d_z^cf(z)-\int_{\partial U_r(s)}f(z)d_z^c\phi(z)\right).$$

Let $s=t\in\mathcal{T}$ be a torsion point. Since $\phi(z)$ remains smooth at torsion points, we get

$$\sum_{t\in\mathcal{T}}\left(\int_{\partial U_r(t)}\phi(z)d_z^cf(z)-\int_{\partial U_r(t)}f(z)d_z^c\phi(z)\right)\xrightarrow[r\to0]{}0. \tag{2.40}$$

Let $s=p\in\mathcal{P}$ be a parabolic fixed point. Recalling from equation (2.24) that $\phi(z)$ admits a $\log\log$-singularity at the parabolic fixed points, we conclude as in the proof of Proposition 2.9.3 that

$$\sum_{p\in\mathcal{P}}\left(\int_{\partial U_r(p)}\phi(z)d_z^cf(z)-\int_{\partial U_r(p)}f(z)d_z^c\phi(z)\right)\xrightarrow[r\to0]{}0. \tag{2.41}$$

Combining (2.40) and (2.41) proves the limit under consideration in (2.39) and hence, the proposition. □

The following theorem is an extension of Theorem 1.11.2 to torsion and parabolic fixed points at the level of currents.

Theorem 2.9.5. *We have the relation of currents on* $\overline{X}$

$$g[\widehat{\mu}_{\mathrm{can}}(z)] = \left(\frac{1}{4\pi}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)[\widehat{\mu}_{\mathrm{hyp}}(z)] + \frac{1}{2}\left[\left(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\right)\widehat{\mu}_{\mathrm{hyp}}(z)\right].$$

Proof. From the equality of differential forms described in Theorem 1.11.2 for all $z\in\overline{X}\backslash\mathcal{S}$, for any smooth function f defined on $\overline{X}$, we have the equation of currents on the compact subset Y_{r_0}

$$g[\widehat{\mu}_{\mathrm{can}}(z)](f) = \left(\frac{1}{4\pi}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)[\widehat{\mu}_{\mathrm{hyp}}(z)](f) + \frac{1}{2}\left[\left(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\right)\widehat{\mu}_{\mathrm{hyp}}(z)\right](f). \tag{2.42}$$

Recalling that

$$\overline{X} = Y_{r_0}\cup U_{r_0},$$

it remains to show that the equality of currents explicated in equation (2.42) also holds true on U_{r_0}.

Combining Propositions 2.9.3 and 2.9.4, for any smooth function f defined on $\overline{X}$, we have the equation of currents on U_{r_0}

$$2g[\widehat{\mu}_{\mathrm{can}}(z)](f) - 2[\widehat{\mu}_{\mathrm{shyp}}(z)](f) = \frac{1}{2\pi}[\widehat{\mu}_{\mathrm{hyp}}(z)](f) + \left[\left(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\right)\widehat{\mu}_{\mathrm{hyp}}(z)\right](f).$$

Solving for $[\widehat{\mu}_{\mathrm{can}}(z)](f)$, we arrive at

$$g[\widehat{\mu}_{\mathrm{can}}(z)](f) = \left(\frac{1}{4\pi}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)[\widehat{\mu}_{\mathrm{hyp}}(z)](f) + \frac{1}{2}\left[\left(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\right)\widehat{\mu}_{\mathrm{hyp}}(z)\right](f),$$

which completes the proof of the theorem. □

Remark 2.9.6. We know that

$$\widehat{\mu}_{\mathrm{hyp}}(z) \text{ and } \left(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\right)\widehat{\mu}_{\mathrm{hyp}}(z),$$

are both singular at the torsion and parabolic fixed points, whereas $\widehat{\mu}_{\mathrm{can}}(z)$ remains smooth at these points. So it is not possible to extend Theorem 1.11.2 to torsion and parabolic fixed points at the level of differential forms. However Theorem 2.9.5 provides an extension of Theorem 1.11.2 to torsion and parabolic fixed points, at the level of currents.

Chapter 3

Key identity for singular functions

From Proposition 2.6.4 proved in last chapter, for $z, w \in X$, we have

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \phi(z) + \phi(w),$$

where

$$\phi(z) = \int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) - \frac{1}{2}\int_X\int_X g_{\mathrm{hyp}}(\xi,\zeta)\,\mu_{\mathrm{can}}(\zeta)\,\mu_{\mathrm{can}}(\xi).$$

Given any smooth function f defined on $\overline{X}$, the key identity proved in Theorem 2.9.5 allows us to express the integral

$$\int_{\overline{X}} f(z)\,\widehat{\mu}_{\mathrm{can}}(z)$$

in terms of the integrals

$$\int_{\overline{X}} f(z)\,\widehat{\mu}_{\mathrm{hyp}}(z) \text{ and } \int_{\overline{X}} f(z)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\bigg)\,\widehat{\mu}_{\mathrm{hyp}}(z).$$

If we can extend Theorem 2.9.5 to functions which have $\log\log$-growth at the parabolic fixed points, and are log-singular at finitely many points of X, using the normalization condition for $g_{\mathrm{hyp}}(z,w)$ from equation (1.30), we find

$$\int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) = \frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;z)dt\bigg)\,\mu_{\mathrm{hyp}}(z).$$

Such an expression will enable us to estimate the function $\phi(z)$ in terms of invariants coming from hyperbolic geometry in coming chapters.

In this chapter, we extend Theorem 2.9.5 to functions which are $\log\log$-singular at the parabolic fixed points, and log-singular at finitely many points of $\overline{X}\backslash\mathcal{P}$.

In Section 3.1, we introduce $C_{\ell,\ell\ell}(\overline{X})$, a space of functions which are $\log\log$-singular at the parabolic fixed points, and log-singular at finitely many points of $\overline{X}\backslash\mathcal{P}$. We then extend Lemma 2.5.4 to functions of $C_{\ell,\ell\ell}(\overline{X})$.

In Section 3.2, following the same strategy and techniques as in Section 2.9, we extend Propositions 2.9.2, 2.9.3, and 2.9.4 to functions of $C_{\ell,\ell\ell}(\overline{X})$. This enables us to extend Theorem 2.9.5 to functions of $C_{\ell,\ell\ell}(\overline{X})$.

3.1 The space $C_{\ell,\ell\ell}(\overline{X})$

In this section, we first define $C_{\ell,\ell\ell}(\overline{X})$, a space of functions which are log log-singular at the parabolic fixed points, and log-singular at finitely many points of $\overline{X}\backslash\mathcal{P}$.

We then prove the existence of a few integrals, whose existence is essential for the extension of Theorem 2.9.5, which is established Section 3.2. We end the section by extending Lemma 2.5.4 to functions of $C_{\ell,\ell\ell}(\overline{X})$.

Definition 3.1.1. Let $C_{\ell,\ell\ell}(\overline{X})$ denote the set of complex-valued functions $f:\overline{X}\rightarrow\mathbb{P}^1(\mathbb{C})$, which admit the following type of singularities at finitely many points $\mathrm{Sing}(f)\subseteq\overline{X}$, and are smooth away from $\mathrm{Sing}(f)$:
(1) If $s\in\mathrm{Sing}(f)\backslash\mathcal{P}$, then as z approaches s, the function f satisfies

$$f(z)=c_{f,s}\log|\vartheta_s(z)|+O_z(1), \tag{3.1}$$

for some $c_{f,s}\in\mathbb{C}$, and the contribution from the term $O_z(1)$ is a smooth function in z.
(2) For $p\in\mathrm{Sing}(f)\cap\mathcal{P}$, as z approaches p, the function f satisfies

$$f(z)=c_{f,p}\log\big(-\log|\vartheta_p(z)|\big)+O_z(1), \tag{3.2}$$

for some $c_{f,p}\in\mathbb{C}$, and the contribution from the term $O_z(1)$ is a smooth function in z.

Remark 3.1.2. For $0<r_0<1$, we find

$$\lim_{\epsilon\rightarrow 0}\int_{\epsilon}^{r_0}\int_0^{2\pi}\frac{r\log\big(-\log r\big)drd\theta}{(r\log r)^2}=\lim_{\epsilon\rightarrow 0}2\pi\int_{\epsilon}^{r_0}\frac{\log\big(-\log r\big)dr}{r(\log r)^2}.$$

Substituting $\rho=-\log r$, we arrive at

$$\begin{aligned}
&\lim_{\epsilon\rightarrow 0}2\pi\int_{\epsilon}^{r_0}\frac{\log(-\log r)dr}{r(\log r)^2}=-2\pi\lim_{\epsilon\rightarrow 0}\int_{-\log\epsilon}^{-\log r_0}\frac{\log\rho\,d\rho}{\rho^2}=\\
&2\pi\lim_{\epsilon\rightarrow 0}\left(\left[\frac{\log\rho}{\rho}\right]_{-\log\epsilon}^{-\log r_0}-\int_{-\log\epsilon}^{-\log r_0}\frac{d\rho}{\rho^2}\right)=\\
&2\pi\lim_{\epsilon\rightarrow 0}\left(-\frac{\log(-\log r_0)}{\log r_0}+\frac{\log(-\log\epsilon)}{\log\epsilon}-\frac{1}{\log r_0}+\frac{1}{\log\epsilon}\right)=\\
&-\frac{2\pi}{\log r_0}\big(\log(-\log r_0)+1\big).
\end{aligned}\tag{3.3}$$

So from equations (3.2) and (3.3), we can deduce that any $f \in C_{\ell,\ell\ell}(\overline{X})$ remains integrable with respect to the hyperbolic metric $\widehat{\mu}_{\mathrm{hyp}}(z)$ at the parabolic fixed points. Moreover, since f admits only finitely many singularities of log-type on $\overline{X}\backslash\mathcal{P}$, we can conclude that the integrals

$$\int_{\overline{X}} f(z)\, \widehat{\mu}_{\mathrm{hyp}}(z), \quad \int_{\overline{X}} f(z)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\, \widehat{K}_{\mathrm{hyp}}(t;z)dt\bigg)\, \widehat{\mu}_{\mathrm{hyp}}(z)$$

exist. Furthermore, the canonical metric $\widehat{\mu}_{\mathrm{can}}(z)$ is a smooth (1,1)-form on $\overline{X}$, which implies the existence of the integral

$$\int_{\overline{X}} f(z)\, \widehat{\mu}_{\mathrm{can}}(z).$$

We use the following notation until the end of this chapter.

Notation 3.1.3. Let $U_{r_0}(s)$ denote an open coordinate disk of fixed radius r_0 around $s \in \mathrm{Sing}(f) \cup \mathcal{S}$, and r_0 is small enough such that any two coordinate disks are disjoint. Put

$$U_{r_0} = \bigcup_{s \in \mathrm{Sing}(f)\cup\mathcal{S}} U_{r_0}(s) \;\text{ and }\; Y_{r_0} = \overline{X}\backslash U_{r_0}.$$

Furthermore, for $0 < r < r_0$, let $U_r(s)$ denote an open coordinate disk of radius r around $s \in \mathrm{Sing}(f) \cup \mathcal{S}$, and let $U_{r_0,r}(s)$ denote the annulus $U_{r_0}(s)\backslash U_r(s)$. Put

$$U_r = \bigcup_{s \in \mathrm{Sing}(f)\cup\mathcal{S}} U_r(s) \;\text{ and }\; U_{r_0,r} = U_{r_0}\backslash U_r.$$

Lemma 3.1.4. *Let $f \in C_{\ell,\ell\ell}(\overline{X})$, then f defines a current of type $(0,0)$ on $\overline{X}$, and furthermore the integral*

$$\int_{\overline{X}} \widehat{g}_{\mathrm{hyp}}(z,w) d_z d_z^c f(z)$$

exists, for a fixed $w \in \overline{X}\backslash(\mathrm{Sing}(f) \cup \mathcal{P})$.

Proof. Since f is log log-singular at the parabolic fixed points, and log-singular at finitely many points of $\overline{X}\backslash\mathcal{P}$, we can conclude that f remains integrable with respect to smooth (1,1)-forms on $\overline{X}$, and hence, defines a current of type (0,0) on $\overline{X}$.

For $z \in \overline{X}$ bounded away from $\mathrm{Sing}(f)$, the (1,1)-form $d_z d_z^c f(z)$ is smooth. So we can conclude that $\widehat{g}_{\mathrm{hyp}}(z,w)$ remains integrable with respect to $d_z d_z^c f(z)$, for $z \in \overline{X}$ bounded away from $\mathrm{Sing}(f)$.

For $s \in \mathrm{Sing}(f)\backslash\mathcal{P}$, we now prove the integrability of $\widehat{g}_{\mathrm{hyp}}(z,w)$ with respect to $d_z d_z^c f(z)$ at s, for a fixed $w \in \overline{X}\backslash(\mathrm{Sing}(f) \cup \mathcal{P})$. Let $U_{r_0}(s)$ and $U_{r_0,r}(s)$ be as defined above in Notation 3.1.3. Then it suffices to show that the integral

$$\int_{U_{r_0}(s)} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) = \lim_{r \to 0} \int_{U_{r_0,r}(s)} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z)$$

exists. For $z \neq s$, we know that $d_z d_z^c \log|\vartheta_s(z)| = 0$. So for $r \neq 0$, from equation (3.1), it follows that $d_z d_z^c f(z)$ defines a smooth form for all $z \in U_{r_0,r}(s)$. Hence, for $r \neq 0$, the existence of the integral

$$\int_{U_{r_0,r}(s)} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z)$$

is known. As f is only log-singular at $s \in \mathrm{Sing}(f) \backslash \mathcal{P}$, the integral

$$\int_{U_{r_0}(s)} f(z)\, \mu_{\mathrm{shyp}}(z) = \lim_{r\to 0} \int_{U_{r_0,r}(s)} f(z)\, \mu_{\mathrm{shyp}}(z)$$

also exists. So it suffices to show that the limit exists

$$\lim_{r\to 0} \bigg(\int_{U_{r_0,r}(s)} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) - \int_{U_{r_0,r}(s)} f(z)\, \mu_{\mathrm{shyp}}(z) \bigg). \tag{3.4}$$

From equation (1.29), we have

$$\int_{U_{r_0,r}(s)} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) - \int_{U_{r_0,r}(s)} f(z)\, \mu_{\mathrm{shyp}}(z) =$$
$$\int_{U_{r_0,r}(s)} g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) - \int_{U_{r_0,r}(s)} f(z) d_z d_z^c g_{\mathrm{hyp}}(z,w).$$

By Stokes's theorem, the right-hand side of the above equation can be expressed as

$$\int_{\partial U_{r_0}(s)} g_{\mathrm{hyp}}(z,w) d_z^c f(z) - \int_{\partial U_{r_0}(s)} f(z) d_z^c g_{\mathrm{hyp}}(z,w) +$$
$$\int_{\partial U_r(s)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \int_{\partial U_r(s)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w)). \tag{3.5}$$

As the first two terms in the above expression do not depend on r, the limit exists

$$\lim_{r\to 0} \bigg(\int_{\partial U_{r_0}(s)} g_{\mathrm{hyp}}(z,w) d_z^c f(z) - \int_{\partial U_{r_0}(s)} f(z) d_z^c g_{\mathrm{hyp}}(z,w) \bigg).$$

So it remains to show that as r approaches zero, the third and fourth terms in (3.5) converge. From equation (3.1) and the fact that $\widehat{g}_{\mathrm{hyp}}(z,w)$ remains smooth for $z \in U_r(s)$, we get

$$\lim_{r\to 0} \bigg(\int_{\partial U_r(s)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \int_{\partial U_r(s)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w)) \bigg) =$$
$$-\lim_{r\to 0} c_{f,s} \bigg(\int_{\partial U_r(s)} g_{\mathrm{hyp}}(z,w) \frac{r}{2} \frac{\partial \log r}{\partial r} \frac{d\theta}{2\pi} + \int_{\partial U_r(s)} \frac{r}{2} \log r \frac{\partial g_{\mathrm{hyp}}(z,w)}{\partial r} \frac{d\theta}{2\pi} \bigg) =$$
$$-\frac{c_{f,s}}{2} g_{\mathrm{hyp}}(s,w),$$

which implies that the limit (3.4) exists. Hence, we can conclude that for a fixed $w \in \overline{X}\backslash(\mathrm{Sing}(f) \cup \mathcal{P})$, the hyperbolic Green's function $\widehat{g}_{\mathrm{hyp}}(z,w)$ remains integrable with respect to $d_z d_z^c f(z)$ at $s \in \mathrm{Sing}(f)\backslash\mathcal{P}$.

From equation (3.2), we know that for $z \in U_{r_0}(p)$, the function $f(z)$ in local coordinates (r,θ), where $z = re^{i\theta}$, is of the form

$$f(z) = c_{f,p} \log\big(-\log|\vartheta_p(z)|\big) + O_z(1) = c_{f,p}\log\big(-\log r\big) + O_r(1).$$

The contribution from the $O_r(1)$ term in the above equation is a smooth function in r. Observing that $d_z d_z^c$ in local coordinates (r,θ) is given by

$$d_z d_z^c = \frac{1}{4\pi}\bigg(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\bigg) r dr d\theta,$$

we compute

$$d_z d_z^c \log\big(-\log r\big) = \frac{1}{4\pi}\Bigg(\frac{\partial^2\log\big(-\log r\big)}{\partial r^2} + \frac{1}{r}\frac{\partial \log\big(-\log r\big)}{\partial r}\Bigg) r dr d\theta =$$
$$\frac{1}{4\pi}\bigg(-\frac{1}{r^2(\log r)^2} - \frac{1}{r^2\log r} + \frac{1}{r^2\log r}\bigg) r dr d\theta = -\frac{1}{4\pi}\frac{r dr d\theta}{(r\log r)^2} =$$
$$-\frac{i}{8\pi}\frac{d\vartheta_p(z)\wedge d\overline{\vartheta}_p(z)}{\big(|\vartheta_p(z)|\log|\vartheta_p(z)|\big)^2} = -\frac{1}{4\pi}\,\mu_{\mathrm{hyp}}(z).$$

So we gather that, for $z \in U_{r_0}(p)$,

$$d_z d_z^c f(z) = -\frac{c_{f,p}}{4\pi}\,\widehat{\mu}_{\mathrm{hyp}}(z) + \eta(z),$$

where $\eta(z)$ is a smooth (1,1)-form on $U_{r_0}(p)$.

The hyperbolic Green's function $\widehat{g}_{\mathrm{hyp}}(z,w)$ is integrable at the parabolic fixed points with respect to $\widehat{\mu}_{\mathrm{hyp}}(z)$ on $\overline{X}$, for a fixed $w \in \overline{X}\backslash\mathcal{P}$. This implies that for $w \in \overline{X}\backslash(\mathrm{Sing}(f)\cup\mathcal{P})$ fixed, $\widehat{g}_{\mathrm{hyp}}(z,w)$ is integrable with respect to $d_z d_z^c f(z)$ at the parabolic fixed points. Hence, we can conclude that the integral

$$\int_{\overline{X}} \widehat{g}_{\mathrm{hyp}}(z,w) d_z d_z^c f(z)$$

exists, for a fixed $w \in \overline{X}\backslash(\mathrm{Sing}(f) \cup \mathcal{P})$. □

Lemma 3.1.5. *For any $f \in C_{\ell,\ell\ell}(\overline{X})$, the integrals*

$$\int_{U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z) d_z d_z^c f(z) =$$
$$\sum_{s\in\mathrm{Sing}(f)\cup\mathcal{S}} \int_{U_{r_0}(s)} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z) d_z d_z^c f(z)$$

exist.

Proof. Since the function $\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)$ is smooth on all of $\overline{X}$, the existence of the above integral follows from arguments similar to the ones used in the proof of Lemma 3.1.4. $\square$

Lemma 3.1.6. *For any $f\in C_{\ell,\ell\ell}(\overline{X})$, the integrals*

$$\int_{U_{r_0}}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d_zd^c_zf(z)=$$
$$\sum_{s\in\mathrm{Sing}(f)\cup\mathcal{S}}\int_{U_{r_0}(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d_zd^c_zf(z)$$

exist.

Proof. From Proposition 2.7.3, we know that for $z\in\overline{X}$ bounded away from the parabolic fixed points, $\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)$ remains smooth, and is log log-singular at the parabolic fixed points, similar to the behavior of $\widehat{g}_{\mathrm{hyp}}(z,w)$ at the parabolic fixed points, for a fixed $w\in\overline{X}\backslash\mathcal{P}$. Hence, for each $s\in\mathrm{Sing}(f)\cup\mathcal{S}$ the existence of the integral

$$\int_{U_{r_0}(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d_zd^c_zf(z)$$

follows from the integrability of $\widehat{g}_{\mathrm{hyp}}(z,w)$ with respect to $d_zd^c_zf(z)$ on $\overline{X}$, for a fixed $w\in\overline{X}\backslash(\mathrm{Sing}(f)\cup\mathcal{P})$, as shown in Lemma 3.1.4. This completes the proof of the lemma. $\square$

The following lemma provides an extension of Lemma 2.5.4 to functions $f\in C_{\ell,\ell\ell}(\overline{X})$.

Lemma 3.1.7. *Let $f\in C_{\ell,\ell\ell}(\overline{X})$, then for $w\in\overline{X}\backslash(\mathrm{Sing}(f)\cup\mathcal{P})$ fixed, we have the equality of integrals*

$$\int_{\overline{X}}\widehat{g}_{\mathrm{hyp}}(z,w)d_zd^c_zf(z)+f(w)+\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}\frac{c_{f,s}}{2}\widehat{g}_{\mathrm{hyp}}(s,w)=\int_{\overline{X}}f(z)\,\widehat{\mu}_{\mathrm{shyp}}(z). \tag{3.6}$$

Proof. Let $w\in\overline{X}\backslash(\mathrm{Sing}(f)\cup\mathcal{P})$ and let $U_r(w)$, $U_r(t)$, $U_r(s)$, and $U_r(p)$ denote open coordinate disks of radius r around w, a torsion point $t\in\mathcal{T}$, $s\in\mathrm{Sing}(f)$, and a parabolic fixed point $p\in\mathcal{P}$, respectively.
Put

$$Y_r=\overline{X}\backslash\Bigg(U_r(w)\cup\bigcup_{\substack{t\in\mathcal{T},t\neq w\\ t\notin\mathrm{Sing}(f)}}U_r(t)\cup\bigcup_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}U_r(s)\cup\bigcup_{p\in\mathcal{P}}U_r(p)\Bigg).$$

From equation (1.29), it suffices to prove that

$$\int_{Y_r} g_{\mathrm{hyp}}(z,w)d_zd_z^c f(z) - \int_{Y_r} f(z)\,\mu_{\mathrm{shyp}}(z) = \int_{Y_r} g_{\mathrm{hyp}}(z,w)d_zd_z^c f(z) -$$
$$\int_{Y_r} f(z)d_zd_z^c g_{\mathrm{hyp}}(z,w) \xrightarrow[r\to 0]{} -f(w) - \sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}} \frac{c_{f,s}}{2} g_{\mathrm{hyp}}(s,w).$$

By Stokes's theorem, we have

$$\int_{Y_r} g_{\mathrm{hyp}}(z,w)d_zd_z^c f(z) - \int_{Y_r} f(z)d_zd_z^c g_{\mathrm{hyp}}(z,w) =$$
$$\int_{\partial U_r(w)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \int_{\partial U_r(w)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w)) +$$
$$\sum_{\substack{t\in\mathcal{T},t\neq w\\ t\notin\mathrm{Sing}(f)}} \left(\int_{\partial U_r(t)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \int_{\partial U_r(t)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w))\right) +$$
$$\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}} \left(\int_{\partial U_r(s)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \int_{\partial U_r(s)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w))\right) +$$
$$\sum_{p\in\mathcal{P}} \left(\int_{\partial U_r(p)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \int_{\partial U_r(p)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w))\right). \quad (3.7)$$

Since $w \notin \mathrm{Sing}(f)$, and f remains smooth on $\overline{X}\backslash\mathrm{Sing}(f)$, using equations (2.14) and (2.15), we derive

$$\int_{\partial U_r(w)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \int_{\partial U_r(w)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w)) \xrightarrow[r\to 0]{} -f(w),$$

and

$$\sum_{\substack{t\in\mathcal{T},t\neq w\\ t\notin\mathrm{Sing}(f)}} \left(\int_{\partial U_r(t)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \int_{\partial U_r(t)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w))\right) \xrightarrow[r\to 0]{} 0,$$

respectively. So it remains to show that, as r approaches zero, the summation of the expressions in the third and fourth lines on the right-hand side of equation (3.7) converges to the quantity

$$-\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}} \frac{c_{f,s}}{2} g_{\mathrm{hyp}}(s,w).$$

From equation (3.1), for $s \in \mathrm{Sing}(f)\backslash\mathcal{P}$, we get

$$\int_{\partial U_r(s)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \int_{\partial U_r(s)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w)) =$$
$$-c_{f,s}\int_{\partial U_r(s)} g_{\mathrm{hyp}}(z,w)\frac{r}{2}\frac{\partial \log r}{\partial r}\frac{d\theta}{2\pi} + c_{f,s}\int_{\partial U_r(s)} \frac{r}{2}\log r\frac{\partial g_{\mathrm{hyp}}(z,w)}{\partial r}\frac{d\theta}{2\pi} + O(r).$$

As $w \notin \mathrm{Sing}(f)$, the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ remains smooth at $s \in \mathrm{Sing}(f)\backslash\mathcal{P}$, so we deduce that

$$\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}\int_{\partial U_r(s)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}\int_{\partial U_r(s)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w)) \xrightarrow[r\to 0]{} -\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}\frac{c_{f,s}}{2}g_{\mathrm{hyp}}(s,w).$$

For any parabolic fixed point $p \in \mathcal{P}$, and h any smooth function on $U_r(p)$, as r approaches zero, we find

$$\int_{\partial U_r(p)} \log\big(-\log r\big) d_z^c h(z) = \int_{\partial U_r(p)} \frac{r}{2}\log\big(-\log r\big)\frac{\partial h(z)}{\partial r}\frac{d\theta}{2\pi} = O(r^\alpha) \tag{3.8}$$

for some $0 < \alpha < 1$, and

$$\begin{aligned}\int_{\partial U_r(p)} h(z) d_z^c \log\big(-\log r\big) &= \int_{\partial U_r(p)} h(z)\frac{r}{2}\frac{\partial \log\big(-\log r\big)}{\partial r}\frac{d\theta}{2\pi}\\ &= \int_{\partial U_r(p)} h(z)\frac{1}{2\log r}\frac{d\theta}{2\pi} = O(1/\log r).\end{aligned} \tag{3.9}$$

Hence, using Corollary 2.4.2, and equations (3.2), (3.8), and (3.9), for $p \in \mathcal{P}$, we compute

$$\begin{aligned}&\int_{\partial U_r(p)} g_{\mathrm{hyp}}(z,w)(-d_z^c f(z)) - \int_{\partial U_r(p)} f(z)(-d_z^c g_{\mathrm{hyp}}(z,w)) =\\ &\frac{4\pi c_{f,p}}{\mathrm{vol}_{\mathrm{hyp}}(X)}\int_{\partial U_r(p)}\frac{r}{2}\log\big(-\log r\big)\frac{\partial\log\big(-\log r\big)}{\partial r}\frac{d\theta}{2\pi} -\\ &\frac{4\pi c_{f,p}}{\mathrm{vol}_{\mathrm{hyp}}(X)}\int_{\partial U_r(p)}\frac{r}{2}\log\big(-\log r\big)\frac{\partial\log\big(-\log r\big)}{\partial r}\frac{d\theta}{2\pi} + O(1/\log r) \xrightarrow[r\to 0]{} 0.\end{aligned}$$

This proves equation (3.6). □

Corollary 3.1.8. *Let $f \in C_{\ell,\ell\ell}(\overline{X})$, then for a fixed $w \in X\backslash(\mathrm{Sing}(f)\cap X)$, we have the equality of integrals*

$$\int_X g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) + f(w) + \sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}\frac{c_{f,s}}{2}g_{\mathrm{hyp}}(s,w) = \int_X f(z)\,\mu_{\mathrm{shyp}}(z).$$

Proof. Since $\widehat{g}_{\mathrm{hyp}}(z,w)$ is integrable with respect to $d_z d_z^c f(z)$, and $f(z)$ is integrable with respect to $\widehat{\mu}_{\mathrm{shyp}}(z)$, and there are only finitely many parabolic fixed points, we have

$$\int_{\overline{X}} \widehat{g}_{\mathrm{hyp}}(z,w) d_z d_z^c f(z) = \int_X g_{\mathrm{hyp}}(z,w) d_z d_z^c f(z), \tag{3.10}$$

$$\int_{\overline{X}} f(z)\,\widehat{\mu}_{\mathrm{shyp}}(z) = \int_X f(z)\,\mu_{\mathrm{shyp}}(z). \tag{3.11}$$

The proof of the corollary follows from Lemma 3.1.7, together with equations (3.10) and (3.11). $\square$

3.2 Key identity for $C_{\ell,\ell\ell}(\overline{X})$

In this section, we extend Propositions 2.9.2, 2.9.3, and 2.9.4 to singular functions $f \in C_{\ell,\ell\ell}(\overline{X})$. We get appropriate residual terms in the extended versions of these propositions, but the residual terms cancel out nicely, to provide an extension of Theorem 2.9.5 to singular functions $f \in C_{\ell,\ell\ell}(\overline{X})$.

We continue to follow the same notation as in Notation 3.1.3. The following proposition is an extension of Proposition 2.9.2 to $C_{\ell,\ell\ell}(\overline{X})$.

Proposition 3.2.1. *Let $f \in C_{\ell,\ell\ell}(\overline{X})$, then we have the equality of integrals*

$$
\begin{aligned}
&-\int_{U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z) d_z d_z^c f(z) = (2g-2)\int_{U_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{can}}(z) + \\
&\int_{\partial U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)) + \\
&\sum_{\substack{s\in \mathrm{Sing}(f)\\ s\notin \mathcal{P}}} \frac{c_{f,s}}{2} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(s).
\end{aligned}
$$

Proof. To prove the proposition, it suffices to prove that

$$
\begin{aligned}
&-\int_{U_{r_0,r}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z) d_z d_z^c f(z) - (2g-2)\int_{U_{r_0,r}} f(z)\,\mu_{\mathrm{can}}(z) \xrightarrow[r\to 0]{} \\
&\int_{\partial U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)) + \\
&\sum_{\substack{s\in \mathrm{Sing}(f)\\ s\notin \mathcal{P}}} \frac{c_{f,s}}{2} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(s).
\end{aligned} \tag{3.12}
$$

From Proposition 2.3.3, for all $z \in U_{r_0,r}$, we have by restriction

$$
-d_z d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z) = (2g-2)\,\mu_{\mathrm{can}}(z).
$$

Using the above relation, the left-hand side of the limit considered in (3.12) simplifies to the expression

$$
\int_{U_{r_0,r}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z d_z^c f(z)) - \int_{U_{r_0,r}} f(z)(-d_z d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)).
$$

By Stokes's theorem, we find that the above expression decomposes into the four terms

$$
\begin{aligned}
&\int_{\partial U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)) + \\
&\int_{\partial U_r} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z) d_z^c f(z) - \int_{\partial U_r} f(z) d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z).
\end{aligned} \tag{3.13}
$$

So it suffices to prove that, as r approaches zero, the expression in the second line of (3.13) converges to the quantity

$$\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}\frac{c_{f,s}}{2}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(s).$$

Recall that the expression in the second line of (3.13) can be written as

$$\sum_{s\in\mathrm{Sing}(f)\cup\mathcal{S}}\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)d^c_z f(z)-$$
$$\sum_{s\in\mathrm{Sing}(f)\cup\mathcal{S}}\int_{\partial U_r(s)}f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z).$$

For each $s\in\mathrm{Sing}(f)\cup\mathcal{S}$, we now investigate the behavior of the limit

$$\lim_{r\to 0}\bigg(\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)d^c_z f(z)-\int_{\partial U_r(s)}f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\bigg).$$

Observe that we have the decomposition

$$U_r=\bigcup_{s\in\mathrm{Sing}(f)\cup\mathcal{S}}U_r(s)=\bigcup_{\substack{t\in\mathcal{T}\\ t\notin\mathrm{Sing}(f)}}U_r(t)\cup\bigcup_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}U_r(s)\cup\bigcup_{p\in\mathcal{P}}U_r(p).\tag{3.14}$$

Since both $\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)$ and f remain smooth on $U_r(t)$, for $t\in\mathcal{T}$ and $t\notin\mathrm{Sing}(f)$, we derive

$$\sum_{\substack{t\in\mathcal{T}\\ t\notin\mathrm{Sing}(f)}}\int_{\partial U_r(t)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)d^c_z f(z)-$$
$$\sum_{\substack{t\in\mathcal{T}\\ t\notin\mathrm{Sing}(f)}}\int_{\partial U_r(t)}f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\xrightarrow[r\to 0]{}0.\tag{3.15}$$

From equation (3.1) and the fact that $\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)$ is smooth on $\overline{X}$, for $s\in\mathrm{Sing}(f)\backslash\mathcal{P}$, we compute

$$\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)d^c_z f(z)=\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\frac{r}{2}\frac{\partial f}{\partial r}\frac{d\theta}{2\pi}=$$
$$c_{f,s}\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\frac{r}{2}\frac{\partial\log r}{\partial r}\frac{d\theta}{2\pi}+O(r)\xrightarrow[r\to 0]{}\frac{c_{f,s}}{2}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(s);$$

and

$$\int_{\partial U_r(s)}f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)=\int_{\partial U_r(s)}f(z)\frac{r}{2}\frac{\partial\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)}{\partial r}\frac{d\theta}{2\pi}=$$
$$c_{f,s}\int_{\partial U_r(s)}\frac{r}{2}\log r\frac{\partial\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)}{\partial r}\frac{d\theta}{2\pi}+O(r)\xrightarrow[r\to 0]{}0.$$

This implies that

$$\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)d^c_zf(z)-$$
$$\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}\int_{\partial U_r(s)}f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\xrightarrow[r\to0]{}\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}\frac{c_{f,s}}{2}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(s). \tag{3.16}$$

From equation (3.2) and the fact that $\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)$ is smooth on $\overline{X}$, for any parabolic fixed point $p\in\mathcal{P}$, we get

$$\int_{\partial U_r(p)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)d^c_zf(z)=\int_{\partial U_r(p)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\frac{r}{2}\frac{\partial f(z)}{\partial r}\frac{d\theta}{2\pi}=$$
$$c_{f,p}\int_{\partial U_r(p)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\frac{r}{2}\frac{\partial\log\big(-\log r\big)}{\partial r}\frac{d\theta}{2\pi}+O(r)=$$
$$c_{f,p}\int_{\partial U_r(p)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\frac{1}{2\log r}\frac{d\theta}{2\pi}+O(r)\xrightarrow[r\to0]{}0;$$

and

$$\int_{\partial U_r(p)}f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)=\int_{\partial U_r(p)}f(z)\frac{r}{2}\frac{\partial\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)}{\partial r}\frac{d\theta}{2\pi}=$$
$$c_{f,p}\int_{\partial U_r(p)}\frac{r}{2}\log\big(-\log r\big)\frac{\partial\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)}{\partial r}\frac{d\theta}{2\pi}+O(r)\xrightarrow[r\to0]{}0.$$

Using the above two equations, we derive

$$\sum_{p\in\mathcal{P}}\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)d^c_zf(z)-$$
$$\sum_{p\in\mathcal{P}}\int_{\partial U_r(s)}f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\xrightarrow[r\to0]{}0. \tag{3.17}$$

Hence, using the decomposition described in equation (3.14), and combining equations (3.15), (3.16), and (3.17), we get

$$\int_{\partial U_r}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)d^c_zf(z)-$$
$$\int_{\partial U_r}f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\xrightarrow[r\to0]{}\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\notin\mathcal{P}}}\frac{c_{f,s}}{2}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(s).$$

This proves the limit under consideration in (3.12), which completes the proof of the proposition. □

The following proposition is an extension of Proposition 2.9.3 to $C_{\ell,\ell\ell}(\overline{X})$.

Proposition 3.2.2. *Let $f \in C_{\ell,\ell\ell}(\overline{X})$, then we have the equality of integrals*

$$-\int_{U_{r_0}} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) d_z d_z^c f(z) = \frac{1}{2\pi} \int_{U_{r_0}} f(z)\, \widehat{\mu}_{\mathrm{hyp}}(z) +$$
$$\int_{U_{r_0}} f(z) \left(\int_0^\infty \Delta_{\mathrm{hyp}} \widehat{K}_{\mathrm{hyp}}(t;z)dt \right) \widehat{\mu}_{\mathrm{hyp}}(z) +$$
$$\int_{\partial U_{r_0}} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)) +$$
$$\sum_{\substack{s \in \mathrm{Sing}(f) \\ s \notin \mathcal{P}}} \frac{c_{f,s}}{2} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s).$$

Proof. To prove the proposition, we have to show that

$$-\int_{U_{r_0,r}} \log \|dz\|^2_{\mathrm{res,hyp}}(z) d_z d_z^c f(z) - \frac{1}{2\pi} \int_{U_{r_0,r}} f(z)\, \mu_{\mathrm{hyp}}(z) -$$
$$\int_{U_{r_0,r}} f(z) \bigg(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z)dt \bigg) \mu_{\mathrm{hyp}}(z) \xrightarrow[r \to 0]{}$$
$$\int_{\partial U_{r_0}} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)) +$$
$$\sum_{\substack{s \in \mathrm{Sing}(f) \\ s \notin \mathcal{P}}} \frac{c_{f,s}}{2} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s). \tag{3.18}$$

From Proposition 2.7.6, for all $z \in U_{r_0,r}$, we have

$$-d_z d_z^c \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) = \frac{1}{2\pi}\, \mu_{\mathrm{hyp}}(z) + \bigg(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z)dt \bigg) \mu_{\mathrm{hyp}}(z).$$

So the left-hand side of the limit considered in (3.18) simplifies to the expression

$$\int_{U_{r_0,r}} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z d_z^c f(z)) - \int_{U_{r_0,r}} f(z)(-d_z d_z^c \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)).$$

By Stokes's theorem, the above expression further decomposes into the four terms

$$\int_{\partial U_{r_0}} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)) +$$
$$\int_{\partial U_r} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) d_z^c f(z) - \int_{\partial U_r} f(z) d_z^c \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z). \tag{3.19}$$

So it suffices to prove that, as r approaches zero, the expression in the second line of (3.19) converges to the quantity

$$\sum_{\substack{s \in \mathrm{Sing}(f) \\ s \notin \mathcal{P}}} \frac{c_{f,s}}{2} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s).$$

Recall that the expression in the second line of (3.19) can be written as

$$\sum_{s\in\mathrm{Sing}(f)\cup\mathcal{P}}\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d^c_z f(z)-$$
$$\sum_{s\in\mathrm{Sing}(f)\cup\mathcal{P}}\int_{\partial U_r(s)} f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z).$$

For each $s\in\mathrm{Sing}(f)\cup\mathcal{P}$, we now investigate the behavior of the limit

$$\lim_{r\to 0}\bigg(\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d^c_z f(z)-\int_{\partial U_r(s)} f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)\bigg).$$

From Remark 2.7.2, we know that $\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)$ and f both remain smooth on $U_r(t)$, for $t\in\mathcal{T}$ and $t\not\in\mathrm{Sing}(f)$, so it follows that

$$\sum_{\substack{t\in\mathcal{T}\\ t\not\in\mathrm{Sing}(f)}}\int_{\partial U_r(t)}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d^c_z f(z)-$$
$$\sum_{\substack{t\in\mathcal{T}\\ t\not\in\mathrm{Sing}(f)}}\int_{\partial U_r(t)} f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)\xrightarrow[r\to 0]{}0. \tag{3.20}$$

From equation (3.1) and the fact that $\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)$ is smooth on $U_r(s)$, for $s\in\mathrm{Sing}(f)\backslash\mathcal{P}$, we compute

$$\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d^c_z f(z)=\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)\frac{r}{2}\frac{\partial f}{\partial r}\frac{d\theta}{2\pi}=$$
$$c_{f,s}\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)\frac{r}{2}\frac{\partial\log r}{\partial r}\frac{d\theta}{2\pi}+O(r)\xrightarrow[r\to 0]{}\frac{c_{f,s}}{2}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s);$$

and

$$\int_{\partial U_r(s)} f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)=\int_{\partial U_r(s)} f(z)\frac{r}{2}\frac{\partial\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)}{\partial r}\frac{d\theta}{2\pi}=$$
$$c_{f,s}\int_{\partial U_r(s)}\frac{r}{2}\log r\frac{\partial\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)}{\partial r}\frac{d\theta}{2\pi}+O(r)\xrightarrow[r\to 0]{}0.$$

This implies that

$$\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\not\in\mathcal{P}}}\int_{\partial U_r(s)}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)d^c_z f(z)-$$
$$\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\not\in\mathcal{P}}}\int_{\partial U_r(s)} f(z)d^c_z\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)\xrightarrow[r\to 0]{}\sum_{\substack{s\in\mathrm{Sing}(f)\\ s\not\in\mathcal{P}}}\frac{c_{f,s}}{2}\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s). \tag{3.21}$$

Using Proposition 2.7.3, and equations (3.2), (3.8), and (3.9), for any parabolic fixed point $p \in \mathcal{P}$, we derive

$$\int_{\partial U_r(p)} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) d^c_z f(z) - \int_{\partial U_r(p)} f(z) d^c_z \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) =$$
$$-\frac{8\pi c_{f,p}}{\mathrm{vol_{hyp}}(X)} \int_{\partial U_r(p)} \frac{r}{2} \log\big(-\log r\big) \frac{\partial \log\big(-\log r\big)}{\partial r} \frac{d\theta}{2\pi} +$$
$$\frac{8\pi c_{f,p}}{\mathrm{vol_{hyp}}(X)} \int_{\partial U_r(p)} \frac{r}{2} \log\big(-\log r\big) \frac{\partial \log\big(-\log r\big)}{\partial r} \frac{d\theta}{2\pi} + O(1/\log r) \xrightarrow[r\to 0]{} 0.$$

From the above equation it follows that

$$\sum_{p\in\mathcal{P}} \int_{\partial U_r(s)} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) d^c_z f(z) -$$
$$\sum_{p\in\mathcal{P}} \int_{\partial U_r(s)} f(z) d^c_z \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) \xrightarrow[r\to 0]{} 0. \tag{3.22}$$

Hence, using the decomposition described in equation (3.14), and combining equations (3.20), (3.21), and (3.22), we get

$$\int_{\partial U_r} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) d^c_z f(z) -$$
$$\int_{\partial U_r} f(z) d^c_z \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) \xrightarrow[r\to 0]{} \sum_{\substack{s\in \mathrm{Sing}(f)\\ s\notin\mathcal{P}}} \frac{c_{f,s}}{2} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s),$$

which completes the proof of the proposition. □

The following proposition is an extension of Proposition 2.9.4 to $C_{\ell,\ell\ell}(\overline{X})$.

Proposition 3.2.3. *Let $f \in C_{\ell,\ell\ell}(\overline{X})$, then we have the equality of integrals*

$$-\int_{U_{r_0}} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) d_z d^c_z f(z) =$$
$$2g \int_{U_{r_0}} f(z)\, \widehat{\mu}_{\mathrm{can}}(z) - 2\int_{U_{r_0}} f(z)\, \widehat{\mu}_{\mathrm{shyp}}(z) +$$
$$\int_{\partial U_{r_0}} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d^c_z f(z)) - \int_{\partial U_{r_0}} f(z)(-d^c_z \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)) +$$
$$\sum_{\substack{s\in \mathrm{Sing}(f)\\ s\notin\mathcal{P}}} \frac{c_{f,s}}{2} \log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s). \tag{3.23}$$

Proof. From Proposition 3.2.1, for f a smooth function on U_{r_0}, we have the relation of integrals on U_{r_0}

$$-\int_{U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z) d_z d_z^c f(z) = (2g-2)\int_{U_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{can}}(z) + \int_{\partial U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)) + \sum_{\substack{s\in \mathrm{Sing}(f)\\ s\notin\mathcal{P}}} \frac{c_{f,s}}{2}\log\|d\vartheta_z\|^2_{\mathrm{res,can}}(s).$$

Subtracting the above equation from the desired equality in (3.23), it follows that, it is sufficient to prove that the difference of the integrals

$$\int_{U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z d_z^c f(z)) - \int_{U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z d_z^c f(z))$$

is equal to the summation of the term

$$2\int_{U_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{can}}(z) - 2\int_{U_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{shyp}}(z) + \int_{\partial U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)(-d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z))$$

with the expression

$$-\int_{\partial U_{r_0}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z^c f(z)) + \int_{\partial U_{r_0}} f(z)(-d_z^c \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)) + \sum_{\substack{s\in \mathrm{Sing}(f)\\ s\notin\mathcal{P}}} \frac{c_{f,s}}{2}\big(\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s) - \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(s)\big).$$

So in order to prove the proposition, it suffices to prove that as r approaches zero, the term

$$\int_{U_{r_0,r}} \log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z)(-d_z d_z^c f(z)) - \int_{U_{r_0,r}} \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)(-d_z d_z^c f(z)) - 2\int_{U_{r_0,r}} f(z)\big(\mu_{\mathrm{can}}(z) - \mu_{\mathrm{shyp}}(z)\big), \tag{3.24}$$

converges to the expression

$$\int_{\partial U_{r_0}} \big(\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) - \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big)(-d_z^c f(z)) - \int_{\partial U_{r_0}} f(z)\big(-d_z^c\big(\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) - \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(z)\big)\big) + \sum_{\substack{s\in \mathrm{Sing}(f)\\ s\notin\mathcal{P}}} \frac{c_{f,s}}{2}\big(\log\|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s) - \log\|d\vartheta_z\|^2_{\mathrm{res,can}}(s)\big).$$

For $z, w \in U_{r_0,r}$, from Proposition 2.6.4, we have

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \phi(z) + \phi(w).$$

Taking $d_z d_z^c$ for $z \in U_{r_0,r}$, we get

$$\mu_{\mathrm{can}}(z) - \mu_{\mathrm{hyp}}(z) = -d_z d_z^c \phi(z). \tag{3.25}$$

Furthermore, for $z \in X$, we have

$$\begin{aligned} &\log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) - \log \|d\vartheta_z\|^2_{\mathrm{res,can}}(z) = \\ &\lim_{w \to z} \big(g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) \big) = 2\phi(z). \end{aligned} \tag{3.26}$$

Hence, using equations (3.25) and (3.26), we find that the expression in (3.24) simplifies to the expression

$$2\int_{U_{r_0,r}} \phi(z)(-d_z d_z^c f(z)) - 2\int_{U_{r_0,r}} f(z)(-d_z d_z^c \phi(z)).$$

By Stokes's theorem, it follows that the above expression decomposes into the expression

$$\begin{aligned} &2\int_{\partial U_{r_0}} \phi(z)(-d_z^c f(z)) - 2\int_{\partial U_{r_0}} f(z)(-d_z^c \phi(z)) + \\ &2\int_{\partial U_r} \phi(z) d_z^c f(z) - 2\int_{\partial U_r} f(z) d_z^c \phi(z). \end{aligned} \tag{3.27}$$

Using equation (3.26), we find that the expression in the first line of (3.27) can be written as

$$\begin{aligned} &\int_{\partial U_{r_0}} \big(\log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) - \log \|d\vartheta_z\|^2_{\mathrm{res,can}}(z) \big)(-d_z^c f(z)) - \\ &\int_{\partial U_{r_0}} f(z)\big(- d_z^c \big(\log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(z) - \log \|d\vartheta_z\|^2_{\mathrm{res,can}}(z) \big) \big). \end{aligned}$$

So it suffices to prove that as r approaches zero, the expression in the second line of (3.27) converges to the limit, i.e.,

$$\begin{aligned} &\int_{\partial U_r} \phi(z) d_z^c f(z) - \int_{\partial U_r} f(z) d_z^c \phi(z) \xrightarrow[r \to 0]{} \\ &\sum_{\substack{s \in \mathrm{Sing}(f) \\ s \notin \mathcal{P}}} \frac{c_{f,s}}{4} \big(\log \|dz\|^2_{\mathrm{res,hyp}}(s) - \log \|dz\|^2_{\mathrm{res,can}}(s) \big). \end{aligned} \tag{3.28}$$

For each $s \in \mathrm{Sing}(f) \cup \mathcal{S}$, we now consider the limit

$$\lim_{r \to 0} \bigg(\int_{\partial U_r(s)} \phi(z) d_z^c f(z) - \int_{\partial U_r(s)} f(z) d_z^c \phi(z) \bigg).$$

Since $\phi(z)$ and $f(z)$ both remain smooth on $U_r(t)$, for $t \in \mathcal{T}$ and $t \notin \mathrm{Sing}(f)$, it follows that

$$\sum_{\substack{t \in \mathcal{T} \\ t \notin \mathrm{Sing}(f)}} \bigg(\int_{\partial U_r(t)} \phi(z) d_z^c f(z) - \int_{\partial U_r(t)} f(z) d_z^c \phi(z) \bigg) \xrightarrow[r \to 0]{} 0. \tag{3.29}$$

From equation (3.1) and the fact that $\phi(z)$ remains smooth on $U_r(s)$, for $s \in \mathrm{Sing}(f) \backslash \mathcal{P}$, we find that

$$\begin{aligned}
&\int_{\partial U_r(s)} \phi(z) d_z^c f(z) - \int_{\partial U_r(s)} f(z) d_z^c \phi(z) = \\
&c_{f,s} \bigg(\int_{\partial U_r(s)} \phi(z) \frac{r}{2} \frac{\partial \log r}{\partial r} \frac{d\theta}{2\pi} - \int_{\partial U_r(s)} \frac{r}{2} \log r \frac{\partial \phi(z)}{\partial r} \frac{d\theta}{2\pi} \bigg) + O(r) \xrightarrow[r \to 0]{} \\
&\frac{c_{f,s}}{2} \phi(s) = \frac{c_{f,s}}{4} \big(\log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s) - \log \|d\vartheta_z\|^2_{\mathrm{res,can}}(s) \big).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\sum_{\substack{s \in \mathrm{Sing}(f) \\ s \notin \mathcal{P}}} \bigg(\int_{\partial U_r(s)} \phi(z) d_z^c f(z) - \int_{\partial U_r(s)} f(z) d_z^c \phi(z) \bigg) \xrightarrow[r \to 0]{} \\
&\sum_{\substack{s \in \mathrm{Sing}(f) \\ s \notin \mathcal{P}}} \frac{c_{f,s}}{4} \big(\log \|d\vartheta_z\|^2_{\mathrm{res,hyp}}(s) - \log \|d\vartheta_z\|^2_{\mathrm{res,can}}(s) \big).
\end{aligned} \tag{3.30}$$

From Corollary 2.6.5 and equations (3.2), (3.8), and (3.9), for a parabolic fixed point $p \in \mathcal{P}$, we compute

$$\begin{aligned}
&\int_{\partial U_r(p)} \phi(z) d_z^c f(z) - \int_{\partial U_r(p)} f(z) d_z^c \phi(z) = \\
&-\frac{4\pi c_{f,p}}{\mathrm{vol_{hyp}}(X)} \int_{\partial U_r(p)} \frac{r}{2} \log\big(-\log r\big) \frac{\partial \log\big(-\log r\big)}{\partial r} \frac{d\theta}{2\pi} + \\
&\frac{4\pi c_{f,p}}{\mathrm{vol_{hyp}}(X)} \int_{\partial U_r(p)} \frac{r}{2} \log\big(-\log r\big) \frac{\partial \log\big(-\log r\big)}{\partial r} \frac{d\theta}{2\pi} + O(1/\log r) \xrightarrow[r \to 0]{} 0.
\end{aligned}$$

Using the above computation, we derive

$$\sum_{p \in \mathcal{P}} \bigg(\int_{\partial U_r(p)} \phi(z) d_z^c f(z) - \int_{\partial U_r(p)} f(z) d_z^c \phi(z) \bigg) \xrightarrow[r \to 0]{} 0. \tag{3.31}$$

Using the decomposition described in equation (3.14), and then combining (3.29), (3.30), and (3.31) proves the limit under consideration in (3.28), which completes the proof of the proposition. □

The following theorem is an extension of Theorem 2.9.5 to $C_{\ell,\ell\ell}(\overline{X})$.

Theorem 3.2.4. *Let $f \in C_{\ell,\ell\ell}(\overline{X})$, then we have the equality of integrals*

$$g\int_{\overline{X}} f(z)\,\widehat{\mu}_{\mathrm{can}}(z) = \left(\frac{1}{4\pi}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\int_{\overline{X}} f(z)\,\widehat{\mu}_{\mathrm{hyp}}(z)+$$
$$\frac{1}{2}\int_{\overline{X}} f(z)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\bigg)\,\widehat{\mu}_{\mathrm{hyp}}(z).$$

Proof. From the equality of differential forms described in Theorem 1.11.2 for all $z \in \overline{X}\backslash\mathcal{S}$, for any $f \in C_{\ell,\ell\ell}(\overline{X})$, we have the equation of integrals on the compact subset Y_{r_0}

$$g\int_{Y_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{can}}(z) = \left(\frac{1}{4\pi}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\int_{Y_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{hyp}}(z)+$$
$$\frac{1}{2}\int_{Y_{r_0}} f(z)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\bigg)\,\widehat{\mu}_{\mathrm{hyp}}(z). \tag{3.32}$$

Recalling that

$$\overline{X} = Y_{r_0} \cup U_{r_0},$$

it remains to show that the equality of integrals explicated in equation (3.32) also holds true on U_{r_0}.

Combining Propositions 3.2.2 and 3.2.3, for any $f \in C_{\ell,\ell\ell}(\overline{X})$, we have the equation of integrals on U_{r_0}

$$\int_{U_{r_0}} \log\|dz\|^2_{\mathrm{res,hyp}}(z)\,(-d_zd_z^c f(z)) = \frac{1}{2\pi}\int_{U_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{hyp}}(z)+$$
$$\int_{U_{r_0}} f(z)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\bigg)\,\widehat{\mu}_{\mathrm{hyp}}(z) =$$
$$2g\int_{U_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{can}}(z) - 2\int_{U_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{shyp}}(z).$$

Solving for

$$g\int_{U_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{can}}(z),$$

we arrive at

$$g\int_{U_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{can}}(z) = \left(\frac{1}{4\pi}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\int_{U_{r_0}} f(z)\,\widehat{\mu}_{\mathrm{hyp}}(z)+$$
$$\frac{1}{2}\int_{U_{r_0}} f(z)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\bigg)\,\widehat{\mu}_{\mathrm{hyp}}(z).$$

This completes the proof of the theorem. □

Corollary 3.2.5. *Let $f \in C_{\ell,\ell\ell}(\overline{X})$, then we have the equality of integrals*

$$g\int_X f(z)\,\mu_{\mathrm{can}}(z) = \bigg(\frac{1}{4\pi}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)\int_X f(z)\,\mu_{\mathrm{hyp}}(z)+ \\ \frac{1}{2}\int_X f(z)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;z)dt\bigg)\,\mu_{\mathrm{hyp}}(z).$$

Proof. Since $f(z)$ is integrable with respect to

$$\widehat{\mu}_{\mathrm{can}}(z),\quad \widehat{\mu}_{\mathrm{hyp}}(z),\quad \text{and}\quad \bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\bigg)\,\widehat{\mu}_{\mathrm{hyp}}(z),$$

and there are only finitely many parabolic fixed points, we can conclude that

$$\int_{\overline{X}} f(z)\,\widehat{\mu}_{\mathrm{can}}(z) = \int_X f(z)\,\mu_{\mathrm{can}}(z), \tag{3.33}$$

$$\int_{\overline{X}} f(z)\,\widehat{\mu}_{\mathrm{hyp}}(z) = \int_X f(z)\,\mu_{\mathrm{hyp}}(z), \tag{3.34}$$

and

$$\int_{\overline{X}} f(z)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,\widehat{K}_{\mathrm{hyp}}(t;z)dt\bigg)\,\widehat{\mu}_{\mathrm{hyp}}(z) = \\ \int_X f(z)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;z)dt\bigg)\,\mu_{\mathrm{hyp}}(z). \tag{3.35}$$

The proof of the corollary follows from Theorem 3.2.4 and equations (3.33), (3.34), and (3.35). □

Using the above corollary we have the following result, which as stated in the beginning of the chapter expresses the difference of the hyperbolic and canonical Green's functions in terms of integrals involving the hyperbolic heat kernel and the hyperbolic metric.

Definition 3.2.6. For notational convenience, we put

$$C_{\mathrm{hyp}} = \int_X\int_X g_{\mathrm{hyp}}(\zeta,\xi)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\times \\ \bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\xi)dt\bigg)\,\mu_{\mathrm{hyp}}(\xi)\,\mu_{\mathrm{hyp}}(\zeta). \tag{3.36}$$

Corollary 3.2.7. *For $z, w \in X$, we have*

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \\ \frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\,\mu_{\mathrm{hyp}}(\zeta)+ \\ \frac{1}{2g}\int_X g_{\mathrm{hyp}}(w,\zeta)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\,\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{4g^2}.$$

Proof. For a fixed $z \in \overline{X} \backslash \mathcal{P}$, the hyperbolic Green's function $\widehat{g}_{\mathrm{hyp}}(z,\zeta) \in C_{\ell,\ell\ell}(\overline{X})$ with $\mathrm{Sing}(\widehat{g}_{\mathrm{hyp}}(z,\cdot)) = \mathcal{P} \cup \{z\}$. So from Corollary 3.2.5, and the normalization condition for $g_{\mathrm{hyp}}(z,\zeta)$ given in equation (1.30), we have

$$\int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) = \frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\,\mu_{\mathrm{hyp}}(\zeta).$$

From which we derive

$$\begin{aligned}\phi(z) &= \int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) - \frac{1}{2}\int_X\int_X g_{\mathrm{hyp}}(\zeta,\xi)\,\mu_{\mathrm{can}}(\xi)\,\mu_{\mathrm{can}}(\zeta)\\ &= \frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\,\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{8g^2}. \qquad (3.37)\end{aligned}$$

Now the proof of the corollary follows directly from Proposition 2.6.4. $\square$

Chapter 4

Convergence of certain automorphic functions

Combining Proposition 2.6.4 and equation (3.37), for $z, w \in X$, we have

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \phi(z) + \phi(w),$$

where

$$\phi(z) = \frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)\left(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;\zeta)dt\right)\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{8g^2}.$$

In order to estimate the difference of the hyperbolic and canonical Green's functions, we need to estimate the function $\phi(z)$. So we first study the integral

$$\int_X g_{\mathrm{hyp}}(z,\zeta)\left(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;\zeta)dt\right)\mu_{\mathrm{hyp}}(\zeta).$$

The computational complexity of the analysis that follows, does not decrease by much on removal of torsion points. So for notational brevity, we omit torsion points for the rest of the thesis. Hence, we assume that the Fuchsian subgroup Γ contains only hyperbolic and parabolic elements.

In this chapter we establish the relation

$$4\pi\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z)dt = \Delta_{\mathrm{hyp}} H(z) + \Delta_{\mathrm{hyp}} P(z),$$

where $H(z)$ and $P(z)$ are certain automorphic functions, which depend only on the hyperbolic and parabolic elements of Γ, respectively. Following the methods from [11] closely, and using the above decomposition and estimates of the hyperbolic Green's function which we later derive in Chapter 5, we obtain bounds for the function $\phi(z)$ in Chapter 6.

The methods and techniques used in this chapter are inspired from the ones used in [14].

In Section 4.1, we introduce the hyperbolic heat trace, the Selberg zeta function and a constant related to it.

In Section 4.2, we show that the automorphic function $P(z)$, and its Laplacian $\Delta_{\mathrm{hyp}} P(z)$ are well defined. We also derive the asymptotics of the function $P(z)$ at the parabolic fixed points.

In Section 4.3, we show that the automorphic function $H(z)$ is well defined. We also ascertain the behavior of the function $H(z)$ at the parabolic fixed points, and show that $\widehat{H}(z)$, its extension to $\overline{X}$, is an element of $C_{\ell,\ell\ell}(\overline{X})$.

We then proceed to show that the following decomposition makes sense

$$4\pi \int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z)dt = \Delta_{\mathrm{hyp}} H(z) + \Delta_{\mathrm{hyp}} P(z).$$

Using the above equality, we obtain an expression for the function $\phi(z)$ in terms of the functions $H(z)$, $P(z)$, and other hyperbolic-geometric invariants.

4.1 A constant related to the Selberg zeta function

In this section, we introduce the Selberg zeta function and a constant related to it. We then define the hyperbolic heat trace, and state the relation between it and the constant related to the Selberg zeta function.

Definition 4.1.1. Let $\mathcal{H}(\Gamma)$ denote a complete set of representatives of non-conjugate, primitive, hyperbolic elements in Γ. The hyperbolic length of the closed geodesic determined by $\gamma \in \mathcal{H}(\Gamma)$ on X is given by

$$\ell_\gamma = \inf\{d_{\mathbb{H}}(z, \gamma z) |\, z \in \mathbb{H}\}.$$

It is well-known that the equality holds true

$$|\mathrm{tr}(\gamma)| = 2\cosh(\ell_\gamma/2),$$

where $\mathrm{tr}(\gamma)$ denotes the trace of the matrix γ.

Definition 4.1.2. The length of the shortest geodesic ℓ_X on X is given by

$$\ell_X = \inf\{d_{\mathbb{H}}(z, \gamma z) |\, \gamma \in \Gamma\backslash\{\mathrm{id}\},\, \gamma\,\text{hyperbolic},\, z \in \mathbb{H}\}.$$

From the definition, it is clear that $\ell_X > 0$.

Definition 4.1.3. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Selberg zeta function associated to X is defined as

$$Z_X(s) = \prod_{\gamma \in \mathcal{H}(\Gamma)} Z_\gamma(s), \quad \text{where } Z_\gamma(s) = \prod_{n=0}^{\infty} \big(1 - e^{(s+n)\ell_\gamma}\big).$$

Definition 4.1.4. The Selberg zeta function $Z_X(s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$, with zeros and poles characterized by the spectral theory of the hyperbolic Laplacian. Furthermore, $Z_X(s)$ has a simple zero at $s = 1$, and the constant

$$c_X = \lim_{s\to 1}\left(\frac{Z_X'(s)}{Z_X(s)} - \frac{1}{s-1}\right), \tag{4.1}$$

is well defined.

Remark 4.1.5. The constant c_X was studied in great detail in [13], and upper and lower bounds for this constant were computed for a hyperbolic Riemann surface of finite volume. The analysis from [13] is extended to certain sequences of compact Riemann surfaces in [12]. We will come back to the bounds for the constant c_X for certain sequences of non-compact Riemann surfaces in detail in Section 7.2.

Definition 4.1.6. For $z \in \mathbb{H}$ and $t \in \mathbb{R}_{\geq 0}$, put

$$HK_{\mathrm{hyp}}(t;z) = \sum_{\substack{\gamma \in \Gamma \backslash \{\mathrm{id}\} \\ \gamma \text{ hyperbolic}}} K_{\mathbb{H}}(t;z,\gamma z).$$

The function $HK_{\mathrm{hyp}}(t;z)$ is invariant under the action of Γ, and hence, defines a function on X.

Similarly, for $z \in \mathbb{H}$ and $t \in \mathbb{R}_{\geq 0}$, the series

$$\sum_{\substack{\gamma \in \Gamma \backslash \{\mathrm{id}\} \\ \gamma \text{ parabolic}}} K_{\mathbb{H}}(t;z,\gamma z),$$

is invariant under the action of Γ, and hence, defines a function on X.

Remark 4.1.7. From the definition of $K_{\mathbb{H}}(t;z,w)$ given by equation (1.10), it is clear that $K_{\mathbb{H}}(t;z,\gamma z)$ is positive for all $\gamma \in \Gamma$, $z \in \mathbb{H}$, and $t \in \mathbb{R}_{\geq 0}$. This implies that $K_{\mathrm{hyp}}(t;z)$ is positive for all $z \in X$ and $t \in \mathbb{R}_{\geq 0}$. So from the inequality

$$\sum_{\substack{\gamma \in \Gamma \backslash \{\mathrm{id}\} \\ \gamma \text{ parabolic}}} K_{\mathbb{H}}(t;z,\gamma z) \leq K_{\mathrm{hyp}}(t;z),$$

we derive that the series

$$\sum_{\substack{\gamma \in \Gamma \backslash \{\mathrm{id}\} \\ \gamma \text{ parabolic}}} K_{\mathbb{H}}(t;z,\gamma z),$$

converges for all $z \in X$ and $t \in \mathbb{R}_{\geq 0}$. Furthermore, we can also make the inference that the function

$$HK_{\mathrm{hyp}}(t;z) = K_{\mathrm{hyp}}(t;z) - K_{\mathbb{H}}(t;0) - \sum_{\substack{\gamma \in \Gamma \backslash \{\mathrm{id}\} \\ \gamma \text{ parabolic}}} K_{\mathbb{H}}(t;z,\gamma z)$$

converges for all $z \in X$ and $t \in \mathbb{R}_{\geq 0}$, and hence, is well defined.

Definition 4.1.8. For $t \in \mathbb{R}_{\geq 0}$, the hyperbolic heat trace $H\mathrm{Tr}K_{\mathrm{hyp}}(t)$ is given by

$$H\mathrm{Tr}K_{\mathrm{hyp}}(t) = \int_X HK_{\mathrm{hyp}}(t;z)\,\mu_{\mathrm{hyp}}(z).$$

The convergence of the integral

$$\int_X HK_{\mathrm{hyp}}(t;z)\,\mu_{\mathrm{hyp}}(z)$$

follows from the Selberg trace formula. Hence, the function $H\mathrm{Tr}K_{\mathrm{hyp}}(t)$ is well defined for all $t \in \mathbb{R}_{\geq 0}$. We refer the reader to Chapter 8 of [6], where the Selberg trace formula has been proved in great detail.

The following lemma gives the relation between c_X, the constant related to the Selberg zeta function, and the hyperbolic heat trace $H\mathrm{Tr}K_{\mathrm{hyp}}(t)$.

Lemma 4.1.9. *The constant c_X can be expressed as*

$$c_X = 1 + \int_0^\infty \big(H\mathrm{Tr}K_{\mathrm{hyp}}(t) - 1\big)dt = \int_0^\infty \big(H\mathrm{Tr}K_{\mathrm{hyp}}(t) - 1 + e^{-t}\big)dt.$$

Proof. We refer the reader to Lemma 4.2 in [13]. □

4.2 Convergence of a certain parabolic series

In this section we prove the absolute and uniform convergence of a certain parabolic series, which we denote by $P(z)$, and derive its asymptotics at the parabolic fixed points.

To prove the convergence of $P(z)$, we follow the same techniques employed in Lemma 5.2 in [14], for proving the convergence of the series

$$\sum_{\substack{\gamma\in\Gamma\backslash\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z).$$

Definition 4.2.1. For $z \in \mathbb{H}$, put

$$P(z) = \sum_{\substack{\gamma\in\Gamma\backslash\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}} g_{\mathbb{H}}(z,\gamma z).$$

The function $P(z)$ is invariant under the action of Γ, and hence, defines a function on X.

Remark 4.2.2. From the proof of Lemma 5.2 in [14], we have the disjoint union decomposition of parabolic elements of Γ

$$\begin{aligned}\{\gamma\in\Gamma\backslash\{\mathrm{id}\},\,\gamma\,\text{parabolic}\} &= \bigcup_{p\in\mathcal{P}}\ \bigcup_{\eta\in\Gamma_p\backslash\Gamma} \big(\eta^{-1}\Gamma_p\eta\backslash\{\mathrm{id}\}\big)\\ &= \bigcup_{p\in\mathcal{P}}\ \bigcup_{\eta\in\Gamma_p\backslash\Gamma}\ \bigcup_{n\neq 0}\{\eta^{-1}\gamma_p^n\eta\},\end{aligned}$$

where γ_p is a generator of the stabilizer subgroup Γ_p of the parabolic fixed point $p \in \mathcal{P}$. This implies that formally, we have

$$\begin{aligned} P(z) = &\sum_{\substack{\gamma\in\Gamma\backslash\{\mathrm{id}\}\\ \gamma\,\text{parabolic}}} g_{\mathbb{H}}(z,\gamma z) = \sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma}\sum_{n\neq 0} g_{\mathbb{H}}(z,\eta^{-1}\gamma_p^n\eta z) \\ =&\sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma}\sum_{n\neq 0} g_{\mathbb{H}}(\eta z,\gamma_p^n\eta z) = \sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma} P_{\mathrm{gen},p}(\eta z), \end{aligned} \tag{4.2}$$

where $P_{\mathrm{gen},p}(z) = \sum_{n\neq 0} g_{\mathbb{H}}(z,\gamma_p^n z)$.

For a parabolic fixed point $p \in \mathcal{P}$, from the definition of the scaling matrix σ_p given in equation (1.1), it follows that

$$P_{\mathrm{gen},p}(z) = \sum_{n\neq 0} g_{\mathbb{H}}(z,\gamma_p^n z) = \sum_{n\neq 0} g_{\mathbb{H}}(\sigma_p^{-1}z,\gamma_\infty^n\sigma_p^{-1}z). \tag{4.3}$$

Lemma 4.2.3. *For $z \in \mathbb{H}$ and $p \in \mathcal{P}$ a parabolic fixed point, the series $P_{\mathrm{gen},p}(z)$ converges absolutely and uniformly, and hence, is well defined on $\mathbb{H}$.*

Proof. For $n \neq 0$, from the definition of the free-space Green's function and that of the matrix γ_∞ given in equations (1.19) and (1.2), respectively, we have

$$g_{\mathbb{H}}(z,\gamma_p^n z) = g_{\mathbb{H}}(\sigma_p^{-1}z,\gamma_\infty^n\sigma_p^{-1}z) = \log\left(\frac{4\,\mathrm{Im}(\sigma_p^{-1}z)^2 + n^2}{n^2}\right) > 0.$$

Since every term of the series $P_{\mathrm{gen},p}(z)$ is real-valued and positive, we have

$$\left|P_{\mathrm{gen},p}(z)\right| = \sum_{n\neq 0}\left|g_{\mathbb{H}}(\sigma_p^{-1}z,\gamma_\infty^n\sigma_p^{-1}z)\right| = P_{\mathrm{gen},p}(z).$$

Furthermore, we have the elementary estimate of the series $P_{\mathrm{gen},p}(z)$

$$\begin{aligned} P_{\mathrm{gen},p}(z) =&\sum_{n\neq 0} g_{\mathbb{H}}(\sigma_p^{-1}z,\gamma_\infty^n\sigma_p^{-1}z) = \sum_{n\neq 0}\log\left(\frac{4\,\mathrm{Im}(\sigma_p^{-1}z)^2 + n^2}{n^2}\right) \\ \leq& 2\log\left(4\,\mathrm{Im}(\sigma_p^{-1}z)^2 + 1\right) + 2\int_1^\infty \log\left(\frac{4\,\mathrm{Im}(\sigma_p^{-1}z)^2 + t^2}{t^2}\right)dt. \end{aligned} \tag{4.4}$$

For a fixed $a \in \mathbb{R}$ and $t \in \mathbb{R}$, consider the function

$$F_a(t) = t\log\left(\frac{a^2}{t^2}+1\right) + \int \frac{2a^2}{a^2+t^2}dt.$$

Observing that

$$\frac{dF_a(t)}{dt} = \log\left(\frac{a^2}{t^2}+1\right) + \frac{t}{a^2/t^2+1}\cdot\left(\frac{-2a^2}{t^3}\right) + \frac{2a^2}{a^2+t^2} = \log\left(\frac{a^2}{t^2}+1\right),$$

we derive

$$\int_1^\infty \log\left(\frac{a^2}{t^2}+1\right)dt = \left[F_a(t)\right]_1^\infty = \left[t\log\left(\frac{a^2}{t^2}+1\right)\right]_1^\infty + \int_1^\infty \frac{2a^2}{a^2+t^2}dt.$$

Substituting $a = 2\operatorname{Im}(\sigma_p^{-1}z)$ in the above equation, we get

$$\begin{aligned}
&\int_1^\infty \log\left(\frac{4\operatorname{Im}(\sigma_p^{-1}z)^2+t^2}{t^2}\right)dt = \\
&\left[t\log\left(\frac{4\operatorname{Im}(\sigma_p^{-1}z)^2+t^2}{t^2}\right)\right]_1^\infty + \int_1^\infty \frac{8\operatorname{Im}(\sigma_p^{-1}z)^2}{4\operatorname{Im}(\sigma_p^{-1}z)^2+t^2}dt = \\
&\left[t\log\left(\frac{4\operatorname{Im}(\sigma_p^{-1}z)^2+t^2}{t^2}\right)\right]_1^\infty + \left[4\operatorname{Im}(\sigma_p^{-1}z)\tan^{-1}\left(\frac{t}{2\operatorname{Im}(\sigma_p^{-1}z)}\right)\right]_1^\infty.
\end{aligned}$$

Evaluating the limits, we get

$$\left[t\log\left(\frac{4\operatorname{Im}(\sigma_p^{-1}z)^2+t^2}{t^2}\right)\right]_1^\infty = -\log\left(4\operatorname{Im}(\sigma_p^{-1}z)^2+1\right) \tag{4.5}$$

and

$$\begin{aligned}
&\left[4\operatorname{Im}(\sigma_p^{-1}z)\tan^{-1}\left(\frac{t}{2\operatorname{Im}(\sigma_p^{-1}z)}\right)\right]_1^\infty = \\
&2\pi\operatorname{Im}(\sigma_p^{-1}z) - 4\operatorname{Im}(\sigma_p^{-1}z)\tan^{-1}\left(\frac{1}{2\operatorname{Im}(\sigma_p^{-1}z)}\right).
\end{aligned} \tag{4.6}$$

Combining equations (4.5) and (4.6), we get

$$\begin{aligned}
&\int_1^\infty \log\left(\frac{4\operatorname{Im}(\sigma_p^{-1}z)^2+t^2}{t^2}\right)dt = \\
&2\pi\operatorname{Im}(\sigma_p^{-1}z) - \log\left(4\operatorname{Im}(\sigma_p^{-1}z)^2+1\right) - 4\operatorname{Im}(\sigma_p^{-1}z)\tan^{-1}\left(\frac{1}{2\operatorname{Im}(\sigma_p^{-1}z)}\right).
\end{aligned} \tag{4.7}$$

Combining inequality (4.4) and equation (4.7), we deduce the inequality

$$P_{\mathrm{gen},p}(z) \leq 4\pi\operatorname{Im}(\sigma_p^{-1}z) - 8\operatorname{Im}(\sigma_p^{-1}z)\tan^{-1}\left(\frac{1}{2\operatorname{Im}(\sigma_p^{-1}z)}\right). \tag{4.8}$$

We now show that the above expression is bounded by $32\operatorname{Im}(\sigma_p^{-1}z)^2$, i.e.,

$$\begin{aligned}
P_{\mathrm{gen},p}(z) &\leq 4\pi\operatorname{Im}(\sigma_p^{-1}z) - 8\operatorname{Im}(\sigma_p^{-1}z)\tan^{-1}\left(\frac{1}{2\operatorname{Im}(\sigma_p^{-1}z)}\right) \\
&\leq 32\operatorname{Im}(\sigma_p^{-1}z)^2,
\end{aligned} \tag{4.9}$$

which will prove the absolute and uniform convergence of $P_{\mathrm{gen},p}(z)$ on $\mathbb{H}$. For this, we consider for $t \in \mathbb{R}_{\geq 0}$, the function

$$F(t) = 32t^2 - 4\pi t + 8t\tan^{-1}\left(\frac{1}{2t}\right).$$

As $F(0) = 0$, to establish inequality (4.9), it suffices to show that the function $F(t)$ is a monotone increasing function for all $t \in \mathbb{R}_{\geq 0}$. So we compute

$$\frac{dF(t)}{dt} = 64t - 4\pi + 8\tan^{-1}\left(\frac{1}{2t}\right) - \frac{16t}{4t^2+1}.$$

To prove that $\frac{dF(t)}{dt} > 0$, for all $t \in \mathbb{R}_{\geq 0}$, it suffices to show that the function

$$G(t) = 64t + 8\tan^{-1}\left(\frac{1}{2t}\right) - \frac{16t}{4t^2+1}$$

is a monotone increasing function for all $t \in \mathbb{R}_{\geq 0}$, as $G(0) = 4\pi$. Computing the derivative of $G(t)$, we find

$$\frac{dG(t)}{dt} = 64 - \frac{16}{4t^2+1} + \frac{64t^2-16}{(4t^2+1)^2} = 64 - \frac{32}{(4t^2+1)^2} > 0,$$

which proves inequality (4.9), and hence, the absolute and uniform convergence of $P_{\mathrm{gen},p}(z)$ on $\mathbb{H}$. □

Using the above lemma, and some of the estimates derived in course of its proof, we prove the absolute and uniform convergence of the automorphic function $P(z)$ in the following proposition.

Proposition 4.2.4. *For $z \in X$, the series $P(z)$ converges absolutely and uniformly, and hence, is well defined on X.*

Proof. For $z \in X$, from equation (4.2), we have

$$P(z) = \sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma} P_{\mathrm{gen},p}(\eta z).$$

From the estimate obtained in equation (4.9), we derive

$$\begin{aligned} P(z) = \sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma} P_{\mathrm{gen},p}(\eta z) &\leq 32\sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma} \mathrm{Im}(\sigma_p^{-1}\eta z)^2 \\ &= 32\sum_{p\in\mathcal{P}} \mathcal{E}_{\mathrm{par},p}(z,2), \end{aligned} \tag{4.10}$$

where $\mathcal{E}_{\mathrm{par},p}(z,2)$ is the parabolic Eisenstein series associated to $p \in \mathcal{P}$ at $s = 2$, as defined in Section 1.5. Since every term of the series $P(z)$ is real-valued and positive, we can conclude that $P(z)$ is absolutely and uniformly convergent on X, and hence, well defined on X. □

In the following proposition, we investigate the behavior of the automorphic function $P(z)$ at the parabolic fixed points.

Proposition 4.2.5. *As $z \in X$ approaches a parabolic fixed point $p \in \mathcal{P}$, the function $P(z)$ satisfies the estimate*

$$P(z) = 4\pi \operatorname{Im}(\sigma_p^{-1}z) - \log\big(4 \operatorname{Im}(\sigma_p^{-1}z)^2\big) + O_z(1),$$

where the contribution from the term $O_z(1)$ is a smooth function in z.

Proof. Let $z \in X$ approach a parabolic fixed point $p \in \mathcal{P}$. From equation (4.2), we obtain the decomposition

$$P(z) = \sum_{\substack{q\in\mathcal{P}\\ q\neq p}} \sum_{\eta\in\Gamma_q\backslash\Gamma} P_{\mathrm{gen},q}(\eta z) + \sum_{\substack{\eta\in\Gamma_p\backslash\Gamma\\ \eta\neq\mathrm{id}}} P_{\mathrm{gen},p}(\eta z) + P_{\mathrm{gen},p}(z). \tag{4.11}$$

We now estimate the right-hand side of the above equation term by term. For the first term, using inequality (4.9), we derive the estimate

$$\sum_{\substack{q\in\mathcal{P}\\ q\neq p}} \sum_{\eta\in\Gamma_q\backslash\Gamma} P_{\mathrm{gen},q}(\eta z) \leq 32 \sum_{\substack{q\in\mathcal{P}\\ q\neq p}} \sum_{\eta\in\Gamma_q\backslash\Gamma} \operatorname{Im}(\sigma_q^{-1}\eta z)^2 = 32 \sum_{\substack{q\in\mathcal{P}\\ q\neq p}} \mathcal{E}_{\mathrm{par},q}(z,2). \tag{4.12}$$

For the second term, we deduce

$$\sum_{\substack{\eta\in\Gamma_p\backslash\Gamma\\ \eta\neq\mathrm{id}}} P_{\mathrm{gen},p}(\eta z) \leq 32 \sum_{\substack{\eta\in\Gamma_p\backslash\Gamma\\ \eta\neq\mathrm{id}}} \operatorname{Im}(\sigma_p^{-1}\eta z)^2 = 32\big(\mathcal{E}_{\mathrm{par},p}(z,2) - \operatorname{Im}(\sigma_p^{-1}z)^2\big). \tag{4.13}$$

To deduce an estimate of (4.12) and (4.13), we use Corollary 1.5.7, which shows that for $z \in X$ approaching $p \in \mathcal{P}$, we have

$$\begin{aligned}\mathcal{E}_{\mathrm{par},q}(z,2) = \delta_{q,p} \operatorname{Im}(\sigma_p^{-1}z)^2 + \alpha_{q,p}(2) \operatorname{Im}(\sigma_p^{-1}z)^{-1} + \\ O\left(\big(1 + \operatorname{Im}(\sigma_p^{-1}z)^{-2}\big) e^{-2\pi \operatorname{Im}(\sigma_p^{-1}z)}\right).\end{aligned}$$

This implies that as $z \in X$ approaches $p \in \mathcal{P}$, we have

$$\sum_{\substack{q\in\mathcal{P}\\ q\neq p}} \sum_{\eta\in\Gamma_q\backslash\Gamma} P_{\mathrm{gen},q}(\eta z) \leq 32 \sum_{\substack{q\in\mathcal{P}\\ q\neq p}} \mathcal{E}_{\mathrm{par},q}(z,2) = O\left(\operatorname{Im}(\sigma_p^{-1}z)^{-1}\right)$$

and

$$\sum_{\substack{\eta\in\Gamma_p\backslash\Gamma\\ \eta\neq\mathrm{id}}} P_{\mathrm{gen},p}(\eta z) \leq 32 \left(\mathcal{E}_{\mathrm{par},p}(z,2) - \operatorname{Im}(\sigma_p^{-1}z)^2\right) = O\left(\operatorname{Im}(\sigma_p^{-1}z)^{-1}\right).$$

Hence, as $z \in X$ approaches $p \in \mathcal{P}$, we arrive at the estimate

$$\sum_{\substack{q\in\mathcal{P}\\ q\neq p}} \sum_{\eta\in\Gamma_q\backslash\Gamma} P_{\mathrm{gen},q}(\eta z) + \sum_{\substack{\eta\in\Gamma_p\backslash\Gamma\\ \eta\neq\mathrm{id}}} P_{\mathrm{gen},p}(\eta z) = O\left(\mathrm{Im}(\sigma_p^{-1}z)^{-1}\right). \tag{4.14}$$

We are now left to investigate the behavior of the third term

$$\begin{aligned} P_{\mathrm{gen},p}(z) &= \sum_{n\neq 0} g_{\mathbb{H}}(\sigma_p^{-1}z, \gamma_\infty^n \sigma_p^{-1}z) \\ &= \lim_{w\to z}\lim_{s\to 1}\bigg(\sum_{n=-\infty}^{\infty} g_{\mathbb{H},s}(\sigma_p^{-1}w, \gamma_\infty^n\sigma_p^{-1}z) - g_{\mathbb{H},s}(\sigma_p^{-1}z, \sigma_p^{-1}w)\bigg), \end{aligned} \tag{4.15}$$

as $z \in X$ approaches $p \in \mathcal{P}$.

From Lemma 5.1 in Chapter 5 of [8], for $\mathrm{Im}(\sigma_p^{-1}z) > \mathrm{Im}(\sigma_p^{-1}w)$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} g_{\mathbb{H},s}(\sigma_p^{-1}w, \gamma_\infty^n\sigma_p^{-1}z) = \frac{4\pi}{2s-1}\,\mathrm{Im}(\sigma_p^{-1}w)^s\,\mathrm{Im}(\sigma_p^{-1}z)^{1-s} + \\ \sum_{n\neq 0}\frac{1}{|n|} W_s(n\sigma_p^{-1}z)\overline{V_{\overline{s}}(n\sigma_p^{-1}w)}. \end{aligned} \tag{4.16}$$

Substituting the above expression in equation (4.15), we get

$$\begin{aligned} &P_{\mathrm{gen},p}(z) = 4\pi\,\mathrm{Im}(\sigma_p^{-1}z) + \\ &\lim_{w\to z}\lim_{s\to 1}\bigg(\sum_{n\neq 0}\frac{1}{|n|}W_s(n\sigma_p^{-1}z)\overline{V_{\overline{s}}(n\sigma_p^{-1}w)} - g_{\mathbb{H},s}(\sigma_p^{-1}z,\sigma_p^{-1}w)\bigg). \end{aligned} \tag{4.17}$$

From the proof of Corollary 1.9.5, we have the estimate

$$\begin{aligned} \sum_{n\neq 0}\frac{1}{|n|}W_s(n\sigma_p^{-1}z)\overline{V_{\overline{s}}(n\sigma_p^{-1}w)} = -\log\left|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\right|^2 + \\ O\Big(e^{-2\pi(\mathrm{Im}(\sigma_p^{-1}z) - \mathrm{Im}(\sigma_p^{-1}w))}\Big). \end{aligned}$$

Using the estimate stated in above equation, we compute

$$\begin{aligned} &\lim_{w\to z}\lim_{s\to 1}\bigg(\sum_{n\neq 0}\frac{1}{|n|}W_s(n\sigma_p^{-1}z)\overline{V_{\overline{s}}(n\sigma_p^{-1}w)} - g_{\mathbb{H},s}(\sigma_p^{-1}z,\sigma_p^{-1}w)\bigg) = \\ &\lim_{w\to z}\bigg(-\log\left|1 - e^{2\pi i(\sigma_p^{-1}z-\sigma_p^{-1}w)}\right|^2 - \log\left|\frac{\sigma_p^{-1}z - \overline{\sigma_p^{-1}w}}{\sigma_p^{-1}z - \sigma_p^{-1}w}\right|^2\bigg) + O_z(1) = \\ &\lim_{w\to z}\log\left|\frac{\sigma_p^{-1}z - \sigma_p^{-1}w}{1 - e^{2\pi i(\sigma_p^{-1}z-\sigma_p^{-1}w)}}\right|^2 - \log\left(4\,\mathrm{Im}(\sigma_p^{-1}z)^2\right) + O_z(1) = \\ &-\log\left(4\,\mathrm{Im}(\sigma_p^{-1}z)^2\right) + O_z(1). \end{aligned} \tag{4.18}$$

Combining equations (4.17) and (4.18), we arrive at the estimate

$$P_{\mathrm{gen},p}(z) = 4\pi \operatorname{Im}(\sigma_p^{-1}z) - \log\big(4\operatorname{Im}(\sigma_p^{-1}z)^2\big) + O_z(1),$$

which along with the estimate obtained in equation (4.14) completes the proof of the proposition. $\square$

Definition 4.2.6. For $z \in X$, we put

$$C'_{\mathrm{par}} = \max_{z\in\mathbb{H}} \Bigg(\sum_{p\in\mathcal{P}} \sum_{\substack{\eta\in\Gamma_p\backslash\Gamma \\ \eta\neq\mathrm{id}}} P_{\mathrm{gen},p}(\eta z) \Bigg). \tag{4.19}$$

The existence of the constant C'_{par} follows from the upper bound derived in equation (4.13).

Remark 4.2.7. While estimating the function $\phi(z)$ later in Chapter 6, we bound some of the quantities involved by the constant C'_{par}.

In the following lemma, we prove the absolute and uniform convergence of $\Delta_{\mathrm{hyp}} P(z)$, the Laplacian applied to the automorphic function $P(z)$.

Lemma 4.2.8. *For $z \in X$, the function $\Delta_{\mathrm{hyp}} P(z)$ converges absolutely and locally uniformly on X, and remains bounded at the parabolic fixed points. Furthermore, we have*

$$\Delta_{\mathrm{hyp}} P(z) = \sum_{p\in\mathcal{P}} \sum_{\eta\in\Gamma_p\backslash\Gamma} \Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\eta z),$$

where $\Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(z) = 2\bigg(\dfrac{2\pi\operatorname{Im}(\sigma_p^{-1}z)}{\sinh(2\pi\operatorname{Im}(\sigma_p^{-1}z))}\bigg)^2 - 2.$

Proof. In Lemma 5.1 of [14], it has been shown that the series

$$\sum_{n\neq 0} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(\sigma_p^{-1}z, \gamma_\infty^n \sigma_p^{-1}z)$$

converges absolutely and uniformly on $\mathbb{H}$. Furthermore, the relation has been established

$$\sum_{n\neq 0} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(\sigma_p^{-1}z, \gamma_\infty^n \sigma_p^{-1}z) = 2\bigg(\frac{2\pi\operatorname{Im}(\sigma_p^{-1}z)}{\sinh(2\pi\operatorname{Im}(\sigma_p^{-1}z))}\bigg)^2 - 2.$$

This implies that

$$\Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(z) = \Delta_{\mathrm{hyp}} \sum_{n\neq 0} g_{\mathbb{H}}(\sigma_p^{-1}z, \gamma_\infty^n \sigma_p^{-1}z) =$$

$$\sum_{n\neq 0} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(\sigma_p^{-1}z, \gamma_\infty^n \sigma_p^{-1}z) = 2\bigg(\frac{2\pi\operatorname{Im}(\sigma_p^{-1}z)}{\sinh(2\pi\operatorname{Im}(\sigma_p^{-1}z))}\bigg)^2 - 2. \tag{4.20}$$

From Proposition 2.8.1, we know that the series

$$\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{parabolic}}}\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)$$

is absolutely and locally uniformly convergent on X, and remains bounded at the parabolic fixed points. Hence, from Proposition 4.2.4, it follows that

$$\Delta_{\mathrm{hyp}}\, P(z)=\Delta_{\mathrm{hyp}}\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{parabolic}}} g_{\mathbb{H}}(z,\gamma z)=\sum_{\substack{\gamma\in\Gamma\setminus\{\mathrm{id}\}\\ \gamma\,\mathrm{parabolic}}}\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z)=$$
$$\sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\setminus\Gamma}\sum_{n\neq 0}\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(\sigma_p^{-1}\eta z,\gamma_\infty^n\sigma_p^{-1}\eta z)=\sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\setminus\Gamma}\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},p}(\eta z).$$

This proves the absolute and locally uniform convergence of the the function $\Delta_{\mathrm{hyp}}\, P(z)$, and its boundedness at the parabolic fixed points. □

Definition 4.2.9. For $z\in X$, we define

$$C''_{\mathrm{par}}=\max_{z\in X}\big|\,\Delta_{\mathrm{hyp}}\, P(z)\big|. \tag{4.21}$$

The existence of the constant C''_{par} follows from Lemma 4.2.8.

In Chapter 6, in course of estimating the function $\phi(z)$, we bound some of the quantities involved by the constant C''_{par}.

Remark 4.2.10. For $z\in\mathbb{H}$, from Proposition 4.2.8, we have

$$\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},p}(z)=2\bigg(\frac{2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}{\sinh(2\pi\,\mathrm{Im}(\sigma_p^{-1}z))}\bigg)^2-2<0.$$

This implies that, for $z\in X$, every term of the series

$$\Delta_{\mathrm{hyp}}\, P(z)=\sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\setminus\Gamma}\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},p}(\eta z)$$

is negative. Hence, we arrive at

$$\big|\,\Delta_{\mathrm{hyp}}\, P(z)\big|=-\sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\setminus\Gamma}\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},p}(\eta z).$$

4.3 Expressing $\phi(z)$ in terms of purely hyperbolic-geometric functions

In this section we introduce another automorphic function $H(z)$, which depends only on the hyperbolic elements of the Fuchsian subgroup Γ. We prove its convergence, and compute the asymptotics of it at the parabolic fixed points.

We end the section with an expression for $\phi(z)$ in terms of $H(z)$, $P(z)$, and other hyperbolic-geometric invariants.

Definition 4.3.1. For $z \in X$, put

$$H(z) = 4\pi \int_0^\infty \left(HK_{\mathrm{hyp}}(t; z) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) dt. \tag{4.22}$$

The function $H(z)$ is invariant under the action of Γ, and hence, defines a function on X.

The following proposition proves the convergence of the function $H(z)$.

Proposition 4.3.2. *The function $H(z)$ is well defined on X. Moreover it satisfies*

$$H(z) = \lim_{w \to z} \big(g_{\mathrm{hyp}}(z, w) - g_{\mathbb{H}}(z, w) \big) - P(z). \tag{4.23}$$

Proof. From Proposition 4.2.4, we know that the series

$$P(z) = \sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma \, \text{parabolic}}} g_{\mathbb{H}}(z, \gamma z) = \sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma \, \text{parabolic}}} 4\pi \int_0^\infty K_{\mathbb{H}}(t; z, \gamma z) dt$$

converges absolutely for all $z \in X$. So, we can interchange summation and integration to obtain

$$P(z) = 4\pi \int_0^\infty \Bigg(\sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma \, \text{parabolic}}} K_{\mathbb{H}}(t; z, \gamma z) \Bigg) dt. \tag{4.24}$$

The integral

$$\int_0^\infty \left(K_{\mathrm{hyp}}(t; z) - K_{\mathbb{H}}(t; 0) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) dt \tag{4.25}$$

converges for all $z \in X$. Using equation (4.24) and the convergence of the integral in (4.25), we can write

$$\begin{aligned} H(z) &= 4\pi \int_0^\infty \left(HK_{\mathrm{hyp}}(t; z) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) dt = \\ 4\pi &\int_0^\infty \Bigg(K_{\mathrm{hyp}}(t; z) - K_{\mathbb{H}}(t; 0) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma \, \text{parabolic}}} K_{\mathbb{H}}(t; z, \gamma z) \Bigg) dt = \\ 4\pi &\int_0^\infty \left(K_{\mathrm{hyp}}(t; z) - K_{\mathbb{H}}(t; 0) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) dt - P(z), \end{aligned} \tag{4.26}$$

which proves the convergence of the function $H(z)$.

We now prove equation (4.23). From the convergence of the integral in (4.25), and an application of Fatou's lemma from real analysis, we can interchange limit and integration in the expression to derive

$$\begin{aligned}&\lim_{w\to z}\big(g_{\mathrm{hyp}}(z,w)-g_{\mathbb{H}}(z,w)\big)=\\&\lim_{w\to z}4\pi\int_0^\infty\bigg(K_{\mathrm{hyp}}(t;z,w)-K_{\mathbb{H}}(t;z,w)-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)dt=\\&4\pi\int_0^\infty\bigg(K_{\mathrm{hyp}}(t;z)-K_{\mathbb{H}}(t;0)-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)dt.\end{aligned}\tag{4.27}$$

Combining equations (4.26) and (4.27) proves equation (4.23). □

In the following proposition, we describe the behavior of the automorphic function $H(z)$ at the parabolic fixed points.

Proposition 4.3.3. *As $z\in X$ approaches a parabolic fixed point $p\in\mathcal{P}$, we have*

$$\begin{aligned}&H(z)=\\&-\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}+4\pi k_{p,p}(0)+O\big(\mathrm{Im}(\sigma_p^{-1}z)^{-1}\big),\end{aligned}$$

where $k_{p,p}(0)$ is the zeroth Fourier coefficient in the Fourier expansion of Kronecker's limit function $\kappa_p(z)$ associated to the parabolic fixed point $p\in\mathcal{P}$ (see Theorem 1.5.3).

Proof. Combining equations (4.23) and (4.11), we have

$$\begin{aligned}&H(z)=\lim_{w\to z}\big(g_{\mathrm{hyp}}(z,w)-g_{\mathbb{H}}(z,w)\big)-P(z)=\\&\lim_{w\to z}\bigg(g_{\mathrm{hyp}}(z,w)-g_{\mathbb{H}}(z,w)-\sum_{n\neq0}g_{\mathbb{H}}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\bigg)-\\&\sum_{\substack{q\in\mathcal{P}\\q\neq p}}\sum_{\eta\in\Gamma_q\backslash\Gamma}P_{\mathrm{gen},q}(\eta z)-\sum_{\substack{\eta\in\Gamma_p\backslash\Gamma\\\eta\neq\mathrm{id}}}P_{\mathrm{gen},p}(\eta z).\end{aligned}$$

We now estimate the right-hand side of the above equation term by term. As $z\in X$ approaches the parabolic fixed point $p\in\mathcal{P}$, from the estimate derived in equation (4.14), we have the estimate of the expression in the second line on the right-hand side of the above equation

$$\sum_{\substack{q\in\mathcal{P}\\q\neq p}}\sum_{\eta\in\Gamma_q\backslash\Gamma}P_{\mathrm{gen},q}(\eta z)+\sum_{\substack{\eta\in\Gamma_p\backslash\Gamma\\\eta\neq\mathrm{id}}}P_{\mathrm{gen},p}(\eta z)=O\left(\mathrm{Im}(\sigma_p^{-1}z)^{-1}\right).$$

So we arrive at the estimate of the function $H(z)$

$$H(z)=\lim_{w\to z}\bigg(g_{\mathrm{hyp}}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H}}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\bigg)+O\big(\mathrm{Im}(\sigma_p^{-1}z)^{-1}\big).\tag{4.28}$$

We are now left to compute the asymptotics of the limit

$$\lim_{w\to z}\bigg(g_{\mathrm{hyp}}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H}}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\bigg)=$$
$$\lim_{w\to z}\lim_{s\to 1}\bigg(g_{\mathrm{hyp},s}(z,w)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)s}\cdot\frac{1}{s-1}-\sum_{n=-\infty}^{\infty}g_{\mathbb{H},s}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\bigg),\tag{4.29}$$

as $z\in X$ approaches the parabolic fixed point $p\in\mathcal{P}$. From the estimate of the automorphic Green's function $g_{\mathrm{hyp},s}(z,w)$ stated in equation (1.25), for $z,w\in X$ with $\mathrm{Im}(\sigma_p^{-1}z)>\mathrm{Im}(\sigma_p^{-1}w)$ and $\mathrm{Im}(\sigma_p^{-1}z)\,\mathrm{Im}(\sigma_p^{-1}w)>C_p^{-2}$, and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$, we have

$$g_{\mathrm{hyp},s}(z,w)=\frac{4\pi\,\mathrm{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},p}(w,s)+$$
$$\sum_{n\neq 0}\frac{1}{|n|}W_s(n\sigma_p^{-1}z)\overline{V_{\overline{s}}(n\sigma_p^{-1}w)}+O\big(e^{-2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}\big).\tag{4.30}$$

So for $z,w\in X$ with $\mathrm{Im}(\sigma_p^{-1}z)>\mathrm{Im}(\sigma_p^{-1}w)$ and $\mathrm{Im}(\sigma_p^{-1}z)\,\mathrm{Im}(\sigma_p^{-1}w)>C_p^{-2}$, and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$, combining equation (4.16) with the estimate stated in equation (4.30), we have

$$g_{\mathrm{hyp},s}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H},s}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)=\frac{4\pi\,\mathrm{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},p}(w,s)-$$
$$\frac{4\pi}{2s-1}\,\mathrm{Im}(\sigma_p^{-1}w)^s\,\mathrm{Im}(\sigma_p^{-1}z)^{1-s}+O\big(e^{-2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}\big).$$

As $z\in X$ approaches $p\in\mathcal{P}$, we can substitute the above estimate in the right-hand side of the limit (4.29) to derive

$$\lim_{w\to z}\bigg(g_{\mathrm{hyp}}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H}}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\bigg)=$$
$$\lim_{w\to z}\lim_{s\to 1}\bigg(\frac{4\pi\,\mathrm{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},p}(w,s)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\cdot\frac{1}{s-1}\bigg)+$$
$$\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}-4\pi\,\mathrm{Im}(\sigma_p^{-1}z)+O\big(e^{-2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}\big).$$

From equation (2.8), as $z\in X$ approaches $p\in\mathcal{P}$, we get

$$\lim_{w\to z}\lim_{s\to 1}\bigg(\frac{4\pi\,\mathrm{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},p}(w,s)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\cdot\frac{1}{s-1}\bigg)=$$
$$\lim_{w\to z}\bigg(4\pi\kappa_p(w)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)-\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)=$$
$$4\pi\kappa_p(z)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)-\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}.$$

So we get

$$\lim_{w\to z}\Bigg(g_{\mathrm{hyp}}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H}}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\Bigg)=4\pi\kappa_p(z)-4\pi\,\mathrm{Im}(\sigma_p^{-1}z)-$$

$$\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}+O\big(e^{-2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}\big). \tag{4.31}$$

From the Fourier expansion of Kronecker's limit function $\kappa_p(z)$ described in Theorem 1.5.3, we have

$$\kappa_p(z)=\mathrm{Im}(\sigma_p^{-1}z)+k_{p,p}(0)-\frac{\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)}+O\big(e^{-2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}\big).$$

Substituting the above estimate of $\kappa_p(z)$ in the expression on the right-hand side of equation (4.31), we get

$$\lim_{w\to z}\Bigg(g_{\mathrm{hyp}}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H}}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\Bigg)=$$

$$-\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}+4\pi k_{p,p}(0)+O\big(e^{-2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}\big). \tag{4.32}$$

Hence, as $z\in X$ approaches $p\in\mathcal{P}$, combining equations (4.28) and (4.32), we arrive at

$$H(z)=$$

$$-\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}+4\pi k_{p,p}(0)+O\big(\mathrm{Im}(\sigma_p^{-1}z)^{-1}\big),$$

which completes the proof of the proposition. □

Corollary 4.3.4. *As $z\in X$ approaches a fixed point $p\in\mathcal{P}$, we have*

$$H(z)=-\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\big(-\log|\vartheta_p(z)|\big)+O_z(1),$$

where the contribution from the term $O_z(1)$ term is a smooth function in z.

Proof. The corollary follows directly from Proposition 4.3.3. □

Definition 4.3.5. From Propositions 4.3.2 and 4.3.3, we know that $H(z)$ is a smooth function for all $z\in X$, and has $\log\log$-growth at the parabolic fixed points. Hence, it defines a singular function $\widehat{H}(z)$ on $\overline{X}$ with a $\log\log$-singularity at the parabolic fixed points, and remains smooth for $z\in\overline{X}\backslash\mathcal{P}$, so $\widehat{H}(z)\in C_{\ell,\ell\ell}(\overline{X})$.

Remark 4.3.6. From Lemma 4.1.9, it follows that

$$\int_X H(z)\,\mu_{\mathrm{hyp}}(z)=4\pi(c_X-1). \tag{4.33}$$

In the following lemma, we obtain a decomposition for the function

$$\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z)dt$$

in terms of the functions $H(z)$ and $P(z)$.

Lemma 4.3.7. *For all $z \in X$, we have*

$$4\pi \int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z)dt = \Delta_{\mathrm{hyp}} H(z) + \Delta_{\mathrm{hyp}} P(z). \tag{4.34}$$

Proof. Using the relation proved in equation (4.23), we get

$$\Delta_{\mathrm{hyp}} P(z) + \Delta_{\mathrm{hyp}} H(z) = \Delta_{\mathrm{hyp}} \lim_{w\to z} \big(g_{\mathrm{hyp}}(z,w) - g_{\mathbb{H}}(z,w)\big).$$

Since the integral

$$4\pi \int_0^\infty \bigg(\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}} K_{\mathbb{H}}(t;z,\gamma z) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \bigg) dt,$$

as well as the integral of the derivatives of the integrand are absolutely convergent, we can take the Laplace operator Δ_{hyp} inside the integral. So we find

$$\begin{aligned} &\Delta_{\mathrm{hyp}} H(z) + \Delta_{\mathrm{hyp}} P(z) = \Delta_{\mathrm{hyp}} \lim_{w\to z} \big(g_{\mathrm{hyp}}(z,w) - g_{\mathbb{H}}(z,w)\big) = \\ &4\pi\ \Delta_{\mathrm{hyp}} \int_0^\infty \bigg(\sum_{\gamma\in\Gamma\setminus\{\mathrm{id}\}} K_{\mathbb{H}}(t;z,\gamma z) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \bigg) dt = \\ &4\pi \int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;z)dt. \end{aligned}$$

This completes the proof of the lemma. □

Theorem 4.3.8. *For $z \in X$, we can express $\phi(z)$ as*

$$\phi(z) = \frac{H(z)}{2g} + \frac{1}{8\pi g}\int_X g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}} P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{8g^2} - \frac{2\pi(c_X-1)}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)}. \tag{4.35}$$

Proof. From equation (3.37), it follows that

$$\begin{aligned} \phi(z) &= \int_X g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) - \frac{1}{2}\int_X\int_X g_{\mathrm{hyp}}(\zeta,\xi)\,\mu_{\mathrm{can}}(\xi)\,\mu_{\mathrm{can}}(\zeta) \\ &= \frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)\left(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;\zeta)dt\right)\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{8g^2}. \end{aligned} \tag{4.36}$$

Furthermore, from equation (4.34), we find

$$\begin{aligned}&\frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)\left(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\right)\mu_{\mathrm{hyp}}(\zeta)=\\&\frac{1}{8\pi g}\int_X g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,H(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)+\frac{1}{8\pi g}\int_X g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta).\end{aligned}\tag{4.37}$$

As $\widehat{H}(\zeta)\in C_{\ell,\ell\ell}(\overline{X})$ with $\mathrm{Sing}(\widehat{H})=\mathcal{P}$, combining Corollary 3.1.8 and equation (4.33), we get

$$\begin{aligned}&\frac{1}{8\pi g}\int_X g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,H(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)=-\frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)d_zd_z^cH(\zeta)=\\&\frac{H(z)}{2g}-\frac{1}{2g}\int_X H(\zeta)\,\mu_{\mathrm{shyp}}(\zeta)=\frac{H(z)}{2g}-\frac{2\pi(c_X-1)}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)}.\end{aligned}\tag{4.38}$$

The proof of the theorem, follows from combining equations (4.36), (4.37), and (4.38). $\square$

Chapter 5

Bounds for heat kernels and hyperbolic Green's functions

In [11], estimates of the hyperbolic heat kernel are derived in terms of certain hyperbolic-geometric invariants on a compact Riemann surface. Using these estimates of the hyperbolic heat kernel, an upper bound was computed for the function $\phi(z)$.

The hyperbolic-geometric invariants used in [11] are not finite for a non-compact Riemann surface. Furthermore, even the function $\phi(z)$ is not bounded on a non-compact Riemann surface. So we restrict ourselves to estimating the function $\phi(z)$ on a compact subset of the non-compact Riemann surface X, which will be done in the next chapter.

In this chapter, we adapt the arguments from [11], to obtain estimates of the hyperbolic heat kernel on a compact subset of the non-compact Riemann surface. We then use the bounds for the heat kernel to derive estimates of the hyperbolic Green's function.

These estimates of the hyperbolic Green's function are used in the next chapter to derive upper bounds for the function $\phi(z)$.

As stated before in Chapter 4, we continue to assume that the Fuchsian subgroup Γ contains only hyperbolic and parabolic elements, i.e., Γ has no torsion elements.

In Section 5.1, we introduce the injectivity radius r_{ε}, and a constant C_{ε}^{HK} associated to the compact subset Y_{ε} of X, which is given by

$$Y_{\varepsilon} = X \backslash \bigcup_{p \in \mathcal{P}} U_{\varepsilon}(p),$$

where $U_{\varepsilon}(p)$ is an open coordinate disk of radius ε around the parabolic fixed point $p \in \mathcal{P}$. We then compute the asymptotics of r_{ε} and C_{ε}^{HK}, as ε approaches zero.

From Section 5.2, we fix an $\varepsilon > 0$ for the rest of the thesis. We then adapt the arguments used in [11] to derive bounds for the heat kernel, and for the hyperbolic Green's function in terms of r_ε and C_ε^{HK} on the compact subset Y_ε of X.

In Section 5.3, we use the estimates derived in Section 5.2, to deduce bounds for the hyperbolic Green's function in the neighborhoods of parabolic fixed points.

5.1 Some hyperbolic-geometric invariants

Let $0 < \varepsilon < 1$ be any number such that the following condition holds true:

$$U_\varepsilon(p) \cap U_\varepsilon(q) = \emptyset \tag{5.1}$$

for all parabolic fixed points $p, q \in \mathcal{P}$ and $p \neq q$, where $U_\varepsilon(p)$, $U_\varepsilon(q)$ denote open coordinate disks of radius ε around $p, q \in \mathcal{P}$, respectively.

Notation 5.1.1. For any $0 < \varepsilon < 1$ satisfying (5.1), put

$$Y_\varepsilon = X \backslash \bigcup_{p \in \mathcal{P}} U_\varepsilon(p).$$

Then, Y_ε is a compact subset of X. For a parabolic fixed point $p \in \mathcal{P}$, put

$$Y_{\varepsilon,p} = X \backslash U_\varepsilon(p), \quad \text{i.e.,} \quad Y_{\varepsilon,p} = Y_\varepsilon \cup \bigcup_{\substack{q \in \mathcal{P} \\ q \neq p}} U_\varepsilon(q).$$

Notation 5.1.2. Let $\mathcal{F} \subset \mathbb{H}$ denote a fixed fundamental domain of the Riemann surface X, and $\Pi : \mathbb{H} \longrightarrow \Gamma\backslash\mathbb{H} = X$ be the universal covering map.

Let Y'_ε, $Y'_{\varepsilon,p}$, and $U'_\varepsilon(p)$ denote $\Pi^{-1}(Y_\varepsilon) \cap \mathcal{F}$, $\Pi^{-1}(Y_{\varepsilon,p}) \cap \mathcal{F}$, and $\Pi^{-1}\big(U_\varepsilon(p)\big) \cap \mathcal{F}$, respectively.

Remark 5.1.3. For a parabolic fixed point $p \in \mathcal{P}$, the set

$$\{\sigma_p^{-1}\eta\mathcal{F} \,|\, \eta \in \Gamma_p\backslash\Gamma\}$$

covers the strip

$$\{z \in \mathbb{H} \,|\, 0 < \mathrm{Re}(z) < 1\}.$$

Similarly, the set

$$\{\sigma_p^{-1}\eta Y'_{\varepsilon,p} \,|\, \eta \in \Gamma_p\backslash\Gamma\}$$

covers the strip

$$\left\{z \in \mathbb{H} \,|\, 0 < \mathrm{Re}(z) < 1,\, 0 < \mathrm{Im}(z) \leq -\frac{\log\varepsilon}{2\pi}\right\}.$$

For the remaining part of this section, we assume that ε satisfies (5.1). We introduce certain hyperbolic-geometric quantities, namely, the injectivity radius r_ε and C_ε^{HK} associated to the compact subset Y_ε of X. We then compute the asymptotics of these two quantities, for a sufficiently small $\varepsilon > 0$.

Definition 5.1.4. We define the injectivity radius r_ε of Y_ε to be

$$r_\varepsilon = \inf\big\{d_{\mathbb{H}}(z,\gamma z) |\, \gamma \in \Gamma\backslash\{\mathrm{id}\},\, z \in Y_\varepsilon'\big\}. \tag{5.2}$$

Remark 5.1.5. As Y_ε is compact, and Γ has no torsion elements, it follows that $r_\varepsilon > 0$.

In the following lemma, we investigate the behavior of the injectivity radius r_ε of the compact subset Y_ε of X, for a sufficiently small $\varepsilon > 0$.

Lemma 5.1.6. *For $\varepsilon > 0$ sufficiently small, we have*

$$\sinh^2(r_\varepsilon/2) = \frac{\pi^2}{(\log\varepsilon)^2}.$$

Proof. From the definition of the injectivity radius, it follows that we need to investigate the behavior of the quantity

$$\inf\big\{d_{\mathbb{H}}(z,\gamma z) |\, \gamma \in \Gamma\backslash\{\mathrm{id}\},\, z \in Y_\varepsilon'\big\}$$

for a sufficiently small $\varepsilon > 0$. Without loss of generality, let us assume that Γ has only one parabolic fixed point, say p with stabilizer subgroup Γ_p. From the definition of the length of the shortest geodesic ℓ_X, we have the lower bound

$$0 < \ell_X \leq \inf\{d_{\mathbb{H}}(z,\gamma z) |\, \gamma \in \Gamma\backslash\{\mathrm{id}\},\, \gamma\,\text{hyperbolic},\, z \in \mathcal{F}\}.$$

As $\varepsilon > 0$ becomes arbitrarily small, any $z \in \partial Y_\varepsilon'$ gets arbitrarily close to the parabolic fixed point, so the infimum of the set

$$\big\{d_{\mathbb{H}}(z,\gamma z) |\, \gamma \in \Gamma\backslash\{\mathrm{id}\},\, \gamma\,\text{parabolic},\, z \in Y_\varepsilon'\big\}$$

also gets arbitrarily small. Hence, for $\varepsilon > 0$ sufficiently small, we can conclude that

$$r_\varepsilon = \inf\big\{d_{\mathbb{H}}(z,\gamma z) |\, \gamma \in \Gamma\backslash\{\mathrm{id}\},\, \gamma\,\text{parabolic},\, z \in Y_\varepsilon'\big\}.$$

As Γ contains only one parabolic fixed point, we have the disjoint union decomposition of parabolic elements of Γ

$$\big\{\gamma \in \Gamma\backslash\{\mathrm{id}\},\, \gamma\,\text{parabolic}\big\} = \bigcup_{\eta \in \Gamma_p\backslash\Gamma} \big\{\eta^{-1}\Gamma_p\eta\backslash\{\mathrm{id}\}\big\}.$$

For $\eta \in \Gamma_p\backslash\Gamma$ and $\eta \neq \mathrm{id}$, we have the lower bound

$$0 < c_\eta \leq \inf\big\{d_{\mathbb{H}}(z,\gamma z) |\, \gamma \in \eta^{-1}\Gamma_p\eta\backslash\{\mathrm{id}\},\, z \in \mathcal{F}\big\},$$

for some constant $c_\eta > 0$. Hence, for $\varepsilon > 0$ sufficiently small, we find

$$r_\varepsilon = \inf\big\{d_{\mathbb{H}}(z,\gamma z)\,|\,\gamma\in\Gamma_p\backslash\{\mathrm{id}\},\, z\in Y'_\varepsilon\big\}.$$

For $z, w \in \mathbb{H}$, from equation (1.3) in [8], we have

$$\cosh(d_{\mathbb{H}}(z,w)) = 1 + 2u(z,w),\ \text{ where } u(z,w) = \frac{|z-w|^2}{4\,\mathrm{Im}(z)\,\mathrm{Im}(w)}. \tag{5.3}$$

Since $\cosh(t)$ is a positive monotone increasing function for $t \in \mathbb{R}_{\geq 0}$, from equation (5.3), for $\varepsilon > 0$ sufficiently small, we have

$$\cosh(r_\varepsilon) = \inf_{\substack{\gamma\in\Gamma_p\backslash\{\mathrm{id}\}\\ z\in Y'_\varepsilon}} \cosh(d_{\mathbb{H}}(z,\gamma z)) = 1 + 2\inf_{\substack{\gamma\in\Gamma_p\backslash\{\mathrm{id}\}\\ z\in Y'_\varepsilon}} u(z,\gamma z).$$

Hence, for $\varepsilon > 0$ sufficiently small, we derive

$$\sinh^2(r_\varepsilon/2) = \inf_{\substack{\gamma\in\Gamma_p\backslash\{\mathrm{id}\}\\ z\in Y'_\varepsilon}} u(z,\gamma z). \tag{5.4}$$

From the definition of the scaling matrix σ_p, for $\gamma\in\Gamma_p\backslash\{\mathrm{id}\}$, we have

$$\sigma_p^{-1}\gamma z = \gamma_\infty^n \sigma_p^{-1} z = \sigma_p^{-1}z + n,$$

for some $n \neq 0$. So using the $\mathrm{PSL}_2(\mathbb{R})$-invariance of the function $u(z,w)$, we compute

$$\inf_{\substack{\gamma\in\Gamma_p\backslash\{\mathrm{id}\}\\ z\in Y'_\varepsilon}} u(z,\gamma z) = \inf_{\substack{\gamma\in\Gamma_p\backslash\{\mathrm{id}\}\\ z\in Y'_\varepsilon}} u(\sigma_p^{-1}z,\sigma_p^{-1}\gamma z) =$$

$$\inf_{\substack{n\in\mathbb{Z}\backslash\{0\}\\ z\in Y'_\varepsilon}} u(\sigma_p^{-1}z,\sigma_p^{-1}z+n) = \inf_{\substack{n\in\mathbb{Z}\backslash\{0\}\\ z\in Y'_\varepsilon}} \frac{n^2}{4\,\mathrm{Im}(\sigma_p^{-1}z)^2} = \frac{\pi^2}{(\log\varepsilon)^2}. \tag{5.5}$$

The proof of the lemma follows from combining equations (5.4) and (5.5). $\square$

Definition 5.1.7. For the remaining part of this chapter, we fix t_0 satisfying $0 < t_0 < 1$, and define

$$C_\varepsilon^{HK} = \max_{z\in Y_\varepsilon}\big(K_{\mathrm{hyp}}(t_0;z)\big). \tag{5.6}$$

As Y_ε is compact, C_ε^{HK} is finite.

In the following lemma we investigate the behavior of C_ε^{HK}, for a sufficiently small $\varepsilon > 0$.

Lemma 5.1.8. *For $\varepsilon > 0$ sufficiently small, we have*

$$C_\varepsilon^{HK} = -\frac{e^{-t_0/4}}{\sqrt{4\pi t_0}}\cdot\frac{\log\varepsilon}{2\pi} + O_z(1),$$

where the contribution from the term $O_z(1)$ is a smooth function in z.

Proof. Without loss of generality, let us assume that Γ has only one parabolic fixed point, say p with stabilizer subgroup Γ_p. For $z \in X$ bounded away from the parabolic fixed point p, the function $K_{\mathrm{hyp}}(t_0; z)$ remains bounded.

The series

$$\sum_{\gamma \in \Gamma \backslash \Gamma_p} K_{\mathbb{H}}(t_0; z, \gamma z)$$

is absolutely and uniformly convergent in a sufficiently small neighborhood of the parabolic fixed point p. So for $z \in X$ approaching p, we can interchange summation and limit to derive

$$\lim_{z \to p} \sum_{\gamma \in \Gamma \backslash \Gamma_p} K_{\mathbb{H}}(t_0; z, \gamma z) = \sum_{\gamma \in \Gamma \backslash \Gamma_p} \lim_{z \to p} K_{\mathbb{H}}(t_0; z, \gamma z).$$

From the integral formula for $K_{\mathbb{H}}(t_0; \rho)$ described in equation (1.10), for every $\gamma \in \Gamma \backslash \Gamma_p$, we have

$$\lim_{z \to p} K_{\mathbb{H}}(t_0; z, \gamma z) = 0,$$

which implies that

$$\lim_{z \to p} \sum_{\gamma \in \Gamma \backslash \Gamma_p} K_{\mathbb{H}}(t_0; z, \gamma z) = 0. \tag{5.7}$$

Furthermore, from Proposition 3.3.5 in [7], which is a reformulation of a result from [20], for $z \in X$ approaching p, we have

$$\begin{aligned} \sum_{\gamma \in \Gamma_p} K_{\mathbb{H}}(t_0; z, \gamma z) &= \frac{e^{-t_0/4} \cdot \mathrm{Im}(\sigma_p^{-1} z)}{\sqrt{4\pi t_0}} + O_z(1) \\ &= -\frac{e^{-t_0/4}}{\sqrt{4\pi t_0}} \cdot \frac{\log|\vartheta_p(z)|}{2\pi} + O_z(1), \end{aligned} \tag{5.8}$$

where the contribution from the term $O_z(1)$ is a smooth function in z. Combining equations (5.7) and (5.8), for $z \in X$ approaching p, we deduce that

$$K_{\mathrm{hyp}}(t_0; z) = -\frac{e^{-t_0/4}}{\sqrt{4\pi t_0}} \cdot \frac{\log|\vartheta_p(z)|}{2\pi} + O_z(1).$$

From the above equation, for $\varepsilon > 0$ sufficiently small, it follows that

$$C_{\varepsilon}^{HK} = -\frac{e^{-t_0/4}}{\sqrt{4\pi t_0}} \cdot \frac{\log \varepsilon}{2\pi} + O_z(1),$$

which completes the proof of the lemma. □

5.2 Bounds for heat kernels and hyperbolic Green's functions

For the remaining part of the thesis, we fix an $0 < \varepsilon < 1$ satisfying (5.1). In this section, we adapt the bounds derived for the hyperbolic heat kernel on a compact Riemann surface in [11] to the compact subset Y_ε of X.

We then use the estimates of the hyperbolic heat kernel to bound the hyperbolic Green's function on the compact subset Y_ε of X.

Lemma 5.2.1. *There exist constants c_0 and c_∞ such that for $0 < t < t_0$, we have*

$$K_{\mathbb{H}}(t;\rho) \leq \frac{c_0}{4\pi t} e^{-\rho^2/(4t)}$$

for all $\rho \geq 0$; and for all $t \geq t_0$, we get

$$K_{\mathbb{H}}(t;\rho) \leq c_\infty e^{-t/4}$$

for all $\rho \geq 0$. Furthermore, there exists a $\delta_0 > 0$, such that $K_{\mathbb{H}}(t;\rho)$ is a monotone decreasing function of ρ for $\rho > \delta_0$ and all $0 < t < t_0$.

Proof. The proof follows directly from the integral formula for $K_{\mathbb{H}}(t;\rho)$ given by equation (1.10). □

Notation 5.2.2. For δ_0 as in Lemma 5.2.1, we fix a δ_ε satisfying

$$\delta_\varepsilon > \max\{\delta_0,\, 4r_\varepsilon + 5\}. \tag{5.9}$$

Definition 5.2.3. For $\delta > 0$ and fixed $z, w \in X$, identifying X with the fundamental domain $\mathcal{F}$, we define the set

$$S_\Gamma(\delta; z, w) = \{\gamma \in \Gamma |\, d_{\mathbb{H}}(z, \gamma w) < \delta\}. \tag{5.10}$$

Remark 5.2.4. From arguments as in the proof of Theorem 2.1 of [15], we have the upper bound

$$\sup_{z,w \in Y_\varepsilon} \#S_\Gamma(\delta; z, w) \leq \frac{\sinh(\delta + r_\varepsilon)}{\sinh(r_\varepsilon)}. \tag{5.11}$$

Definition 5.2.5. For any $\delta \geq \delta_\varepsilon$, $\alpha > 0$, and $z, w \in Y_\varepsilon$, put

$$K_{\mathrm{hyp}}^{\alpha,\delta}(t; z, w) =$$
$$K_{\mathrm{hyp}}(t; z, w) - \sum_{n\colon 0 \leq \lambda_n < \alpha} \varphi_n(z)\varphi_n(w) e^{-\lambda_n t} - \sum_{\gamma \in S_\Gamma(\delta; z, w)} K_{\mathbb{H}}(t; d_{\mathbb{H}}(z, \gamma w)),$$

where $\{\lambda_n\}$ denotes the set of discrete eigenvalues of the hyperbolic Laplacian Δ_{hyp} with associated orthonormal eigenfunctions $\{\varphi_n(z)\}$ (see also equation (1.15)).

Lemma 5.2.6. *For any $\delta > 0$, $t \in \mathbb{R}_{>0}$, and $z, w \in Y_\varepsilon$, we have the upper bound*

$$\sum_{\gamma \in S_\Gamma(\delta; z, w)} K_{\mathbb{H}}(t; d_{\mathbb{H}}(z, \gamma w)) \leq K_{\mathrm{hyp}}(t; z, w). \tag{5.12}$$

Furthermore, for all $0 < t < t_0$ and $\delta > \delta_0$, we have the upper bound

$$K_{\mathrm{hyp}}(t; z, w) \leq \sum_{\gamma \in S_\Gamma(\delta; z, w)} K_{\mathbb{H}}(t; d_{\mathbb{H}}(z, \gamma w)) + \frac{\sinh(r_\varepsilon)\sinh(\delta)}{\sinh^2(r_\varepsilon/2)} \cdot K_{\mathbb{H}}(t; \delta) + \frac{1}{\sinh^2(r_\varepsilon/2)} \int_{\delta - 4r_\varepsilon}^{\infty} K_{\mathbb{H}}(t; \rho) \sinh(\rho + 2r_\varepsilon) d\rho. \tag{5.13}$$

Proof. We refer the reader to [15], where inequalities (5.12) and (5.13) have been established in course of the proof of Theorem 2.1. □

Remark 5.2.7. The eigenfunctions $\{\varphi_n(z)\}$ of the hyperbolic Laplacian Δ_{hyp} can all be chosen to be real-valued. So for the remaining part of the thesis, we assume that the eigenfunctions $\{\varphi_n(z)\}$ are real-valued.

The following lemma is an adaption of Lemma 4.1 proved in [11] to the compact subset Y_ε of X.

Lemma 5.2.8. *Let t_0 and C_ε^{HK} be as in Section 5.1. For any $\alpha > 0$ and $z, w \in Y_\varepsilon$, we have the upper bound involving the eigenfunctions $\varphi_n(z)$ of the hyperbolic Laplacian Δ_{hyp}*

$$\sum_{n \colon 0 \leq \lambda_n < \alpha} |\varphi_n(z)\varphi_n(w)| \leq C_\varepsilon^{HK} e^{\alpha t_0}.$$

Proof. From the estimate

$$|\varphi_n(z)\varphi_n(w)| \leq \frac{1}{2}\left(\varphi_n^2(z) + \varphi_n^2(w)\right),$$

if follows that for $z, w \in Y_\varepsilon$, it suffices to prove that

$$\sum_{n \colon 0 \leq \lambda_n < \alpha} \varphi_n^2(z) \leq C_\varepsilon^{HK} e^{\alpha t_0}.$$

From the spectral expansion of the hyperbolic heat kernel given by equation (1.15), we have

$$K_{\mathrm{hyp}}(t_0; z) = \sum_{n=0}^{\infty} \varphi_n^2(z) e^{-\lambda_n t_0} + \frac{1}{4\pi} \sum_{p \in \mathcal{P}} \int_{-\infty}^{\infty} \left|\mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir)\right|^2 e^{-(r^2 + 1/4)t_0} dr.$$

Observe that the functions

$$\varphi_n^2(z) e^{-\lambda_n t_0} \quad \text{and} \quad \int_{-\infty}^{\infty} \left|\mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir)\right|^2 e^{-(r^2 + 1/4)t_0} dr$$

both remain positive for all $z \in Y_\varepsilon$, $n \in \mathbb{N}$, and $p \in \mathcal{P}$. So for $z \in Y_\varepsilon$, we derive the estimate

$$\sum_{n:\, 0 \le \lambda_n < \alpha} \varphi_n^2(z) e^{-\lambda_n t_0} \le K_{\mathrm{hyp}}(t_0; z) \le C_\varepsilon^{HK}. \tag{5.14}$$

For all $0 \le \lambda_n < \alpha$, we have

$$e^{-\lambda_n t_0} e^{\alpha t_0} \ge 1.$$

Hence, for $z \in Y_\varepsilon$, using the estimate derived in (5.14), we deduce that

$$\begin{aligned} \sum_{n:\, 0 \le \lambda_n < \alpha} \varphi_n^2(z) \, &\le \sum_{n:\, 0 \le \lambda_n < \alpha} \varphi_n^2(z) e^{-\lambda_n t_0} e^{\alpha t_0} \\ &\le \, K_{\mathrm{hyp}}(t_0; z) e^{\alpha t_0} \le C_\varepsilon^{HK} e^{\alpha t_0}. \end{aligned}$$

This completes the proof of the lemma. □

The following lemma is an adaption of Lemma 4.2 proved in [11] to the compact subset Y_ε of X.

Lemma 5.2.9. *Let t_0, c_0, c_∞, r_ε, δ_ε, and C_ε^{HK} be as in Sections 5.1 and 5.2. For any $\delta \ge \delta_\varepsilon$, $\alpha > 0$, and $z, w \in Y_\varepsilon$, we have the following upper bounds:*
(a) If $0 < t < t_0$, then

$$\left| K_{\mathrm{hyp}}^{\alpha,\delta}(t; z, w) \right| \le C_\varepsilon^{HK} e^{\alpha t_0} + \frac{c_0 \sinh(r_\varepsilon) \sinh(\delta)}{8\delta^2 \sinh^2(r_\varepsilon/2)} + \frac{c_0 e^{2r_\varepsilon}}{2\pi \sinh^2(r_\varepsilon/2)};$$

(b) If $t \ge t_0$, then

$$\left| K_{\mathrm{hyp}}^{\alpha,\delta}(t; z, w) \right| \le C_\varepsilon^{HK} e^{-\alpha(t - t_0)} + \frac{c_\infty \sinh(\delta + r_\varepsilon)}{\sinh(r_\varepsilon)} e^{-t/4}.$$

Proof. This result has been established in Lemma 4.2 of [11], when X does not admit parabolic fixed points. The estimate stated in (5.11), and Lemmas 5.2.1, 5.2.6, and 5.2.8 ensure that the proof of Lemma 4.2 of [11], extends to the compact subset Y_ε of X. □

The following lemma is a slight refinement of the bound obtained in Lemma 4.4 in [11].

Lemma 5.2.10. *For $z, w \in \mathbb{H}$ with $0 < a \le d_{\mathbb{H}}(z, w)$, we have the bounds*

$$0 < g_{\mathbb{H}}(z, w) \le -\log(\tanh^2(a/2)).$$

Proof. Recall that for $z, w \in \mathbb{H}$ and $d_{\mathbb{H}}(z, w) > 0$, we have

$$g_{\mathbb{H}}(z, w) = -\log \left| \frac{z - w}{z - \overline{w}} \right|^2.$$

From the above equation, for $z, w \in \mathbb{H}$ and $d_{\mathbb{H}}(z,w) > 0$, it follows that $g_{\mathbb{H}}(z,w)$ is real-valued and strictly positive.

From p. 130 in [3], we have

$$g_{\mathbb{H}}(z,w) = -\log\big(\tanh^2(d_{\mathbb{H}}(z,w)/2)\big).$$

For $t \in \mathbb{R}_{\geq 0}$, the function $\tanh^2(t)$ is a monotone increasing function satisfying the condition

$$0 \leq \tanh^2(t) \leq 1.$$

Furthermore, $\log t$ is also a monotone increasing function for $t \in \mathbb{R}_{\geq 0}$, so for $z, w \in \mathbb{H}$ with $0 < a \leq d_{\mathbb{H}}(z,w)$, we have

$$-\infty < \log\big(\tanh^2(a/2)\big) \leq \log\big(\tanh^2(d_{\mathbb{H}}(z,w)/2)\big) \leq 0.$$

This implies that

$$g_{\mathbb{H}}(z,w) = -\log\big(\tanh^2(d_{\mathbb{H}}(z,w)/2)\big) \leq -\log\big(\tanh^2(a/2)\big),$$

which completes the proof of the lemma. $\square$

The following theorem has been proved as Theorem 4.5 in [11], when X does not admit parabolic fixed points. Lemma 5.2.9 and the estimate stated in equation (5.11) ensure that the computations carried out in the proof of Theorem 4.5 of [11] extend to the compact subset Y_ε of X.

Theorem 5.2.11. *Let t_0, c_0, c_∞, r_ε, δ_ε, and C_ε^{HK} be as in Sections 5.1 and 5.2. For any $\alpha > 0$, $\delta > 0$, and $z, w \in Y_\varepsilon$, we have the upper bound*

$$\Bigg| g_{\mathrm{hyp}}(z,w) - \sum_{n\colon 0<\lambda_n<\alpha} \frac{4\pi}{\lambda_n}\varphi_n(z)\varphi_n(w) - \sum_{\gamma\in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Bigg| \leq B_{\varepsilon,\alpha,\delta},$$

where, for $\delta > \delta_\varepsilon$, we have

$$B_{\varepsilon,\alpha,\delta} = 4\pi\Bigg(C_\varepsilon^{HK} e^{\alpha t_0} + \frac{c_0\sinh(r_\varepsilon)\sinh(\delta)}{8\delta^2\sinh^2(r_\varepsilon/2)} + \frac{c_0 e^{2r_\varepsilon}}{2\pi\sinh^2(r_\varepsilon/2)} + \frac{4c_\infty\sinh(\delta+r_\varepsilon)}{\sinh(r_\varepsilon)} + \frac{C_\varepsilon^{HK}}{\alpha}\Bigg);$$

and for $\delta \leq \delta_\varepsilon$, we have

$$B_{\varepsilon,\alpha,\delta} = B_{\varepsilon,\alpha,\delta_\varepsilon} + \frac{\sinh(\delta_\varepsilon + r_\varepsilon)}{\sinh(r_\varepsilon)}\big|\log\big(\tanh^2(\delta/2)\big)\big|.$$

Proof. The proof of the theorem follows from Theorem 4.5 of [11], after considering the refined estimate obtained in Lemma 5.2.10. $\square$

In the following corollary, we derive an upper bound for the hyperbolic Green's function, when $z \in Y_\varepsilon$ and $w \in \partial Y_{\varepsilon/2}$. This upper bound will be quite useful for computing an estimate of the function $\phi(z)$ on the compact subset Y_ε of X, which will be done in the next chapter. For this purpose we make the following definition.

Definition 5.2.12. Put

$$c_\varepsilon = \inf\big\{ d_{\mathbb{H}}(z, \gamma w) \,|\, \gamma \in \Gamma,\, z \in Y'_\varepsilon,\, w \in \partial Y'_{\varepsilon/2} \big\}. \tag{5.15}$$

From the definition of the constant c_ε, it is clear that $c_\varepsilon > 0$.

Corollary 5.2.13. *For any $\alpha \in (0, \lambda_1)$, $\delta \in (0, c_\varepsilon)$, $z \in Y_\varepsilon$, and $w \in \partial Y_{\varepsilon/2}$, we have the upper bound*

$$|g_{\mathrm{hyp}}(z,w)| \leq B_{\varepsilon/2,\alpha,\delta},$$

where λ_1 is the first non-zero eigenvalue of the hyperbolic Laplacian Δ_{hyp}, and c_ε is as defined in (5.15).

Proof. For any $\alpha \in (0, \lambda_1)$, $\delta \in (0, c_\varepsilon)$, $z \in Y_\varepsilon$, and $w \in \partial Y_{\varepsilon/2}$, we have

$$\big|g_{\mathrm{hyp}}(z,w)\big| = \bigg| g_{\mathrm{hyp}}(z,w) - \sum_{n\colon 0<\lambda_n<\alpha} \frac{4\pi}{\lambda_n}\varphi_n(z)\varphi_n(w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) \bigg|.$$

The proof of the corollary follows directly from Theorem 5.2.11. □

5.3 Bounds for the hyperbolic Green's function at parabolic fixed points

In this section, using the estimates from the previous section, we derive estimates of the hyperbolic Green's function in the neighborhoods of parabolic fixed points. The estimates obtained in this section will enable us to compute estimates of the canonical Green's function in the neighborhoods of parabolic fixed points.

Lemma 5.3.1. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, for a fixed $z \in Y_\varepsilon$ and $w \in U_\varepsilon(p)$, we have the relation*

$$g_{\mathrm{hyp}}(z,w) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(w)|}{\log \varepsilon} \right) + g_p(z,w),$$

where $g_p(z,w)$ is a harmonic function in the variable $w \in U_\varepsilon(p)$.

Proof. For a fixed $z \in Y_\varepsilon$ and $w \in U_\varepsilon(p)$, the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ is a solution of the differential equation

$$d_w d_w^c u(w) = \mu_{\mathrm{shyp}}(w) = \frac{\mu_{\mathrm{hyp}}(w)}{\mathrm{vol}_{\mathrm{hyp}}(X)}. \tag{5.16}$$

Notice that the function

$$-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(w)|}{\log\varepsilon}\right)$$

also satisfies the differential equation (5.16) in the neighborhood $U_\varepsilon(p)$. This implies that, for a fixed $z\in Y_\varepsilon$ and $w\in U_\varepsilon(p)$, we find

$$g_{\mathrm{hyp}}(z,w)=-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(w)|}{\log\varepsilon}\right)+g_p(z,w),$$

where $g_p(z,w)$ is a harmonic function in the variable $w\in U_\varepsilon(p)$. This completes the proof of the lemma. $\square$

Corollary 5.3.2. *Let $p\in\mathcal{P}$ be a parabolic fixed point. Then, for any $\alpha\in(0,\lambda_1)$ and $\delta\in(0,c_\varepsilon)$, we have the upper bound*

$$\sup_{\substack{z\in Y_\varepsilon\\ w\in U_{\varepsilon/2}(p)}}\big|g_p(z,w)\big|\leq B_{\varepsilon/2,\alpha,\delta},$$

where c_ε is as defined in (5.15).

Proof. From Lemma 5.3.1, for a fixed $z\in Y_\varepsilon$ and $w\in U_{\varepsilon/2}(p)$, we have

$$g_{\mathrm{hyp}}(z,w)=-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(w)|}{\log(\varepsilon/2)}\right)+g_p(z,w).$$

From the construction of the function $g_p(z,w)$, it follows that for a fixed $z\in Y_\varepsilon$ and $w\in\partial U_{\varepsilon/2}(p)$, we have

$$g_p(z,w)=g_{\mathrm{hyp}}(z,w).$$

As $g_p(z,w)$ is a harmonic function, $|g_p(z,w)|$ is a subharmonic function. So from the maximum principle for subharmonic functions, and Corollary 5.2.13, for a fixed $z\in Y_\varepsilon$, we deduce the upper bound

$$\begin{aligned}&\sup_{w\in U_{\varepsilon/2}(p)}\big|g_p(z,w)\big|=\sup_{w\in\partial U_{\varepsilon/2}(p)}\big|g_p(z,w)\big|=\\&\sup_{w\in\partial U_{\varepsilon/2}(p)}\big|g_{\mathrm{hyp}}(z,w)\big|\leq\sup_{w\in\partial Y_{\varepsilon/2}}\big|g_{\mathrm{hyp}}(z,w)\big|\leq B_{\varepsilon/2,\alpha,\delta},\end{aligned}$$

for any $\alpha\in(0,\lambda_1)$ and $\delta\in(0,c_\varepsilon)$. The proof of the corollary follows from the fact that the upper bound derived above does not depend on the fixed $z\in Y_\varepsilon$. $\square$

Lemma 5.3.3. *Let $p\in\mathcal{P}$ be a parabolic fixed point. Then, for any $\delta>0$, $z\in U_\varepsilon(p)$, and a fixed $w\in Y_\varepsilon$, we have the relation*

$$g_{\mathrm{hyp}}(z,w)-\sum_{\gamma\in S_\Gamma(\delta;z,w)}g_{\mathbb{H}}(z,\gamma w)=-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right)+h_{\delta,p}(z,w),$$

where $h_{\delta,p}(z,w)$ is a harmonic function in the variable $z\in U_\varepsilon(p)$.

Proof. For any $\delta > 0$, $z \in U_\varepsilon(p)$, and a fixed $w \in Y_\varepsilon$, both functions

$$g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\,, \qquad -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right)$$

are solutions of the differential equation

$$d_z d_z^c u(z) = \mu_{\mathrm{shyp}}(z) = \frac{\mu_{\mathrm{hyp}}(z)}{\mathrm{vol}_{\mathrm{hyp}}(X)}.$$

So we find that

$$g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right) + h_{\delta,p}(z,w),$$

where $h_{\delta,p}(z,w)$ is a harmonic function in the variable $z \in U_\varepsilon(p)$. □

Corollary 5.3.4. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, for any $\alpha \in (0,\lambda_1)$ and $\delta > 0$, we have the upper bound*

$$\sup_{\substack{z \in U_\varepsilon(p)\\ w \in Y_\varepsilon}} \big|h_{\delta,p}(z,w)\big| \leq B_{\varepsilon,\alpha,\delta}.$$

Proof. From Lemma 5.3.3, for any $\delta > 0$, $z \in U_\varepsilon(p)$, and a fixed $w \in Y_\varepsilon$, we have

$$g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right) + h_{\delta,p}(z,w).$$

From the construction of the function $h_{\delta,p}(z,w)$, it follows that for any $\delta > 0$, $z \in \partial U_\varepsilon(p)$, and a fixed $w \in Y_\varepsilon$, we have

$$h_{\delta,p}(z,w) = g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w).$$

As $h_{\delta,p}(z,w)$ is a harmonic function, $|h_{\delta,p}(z,w)|$ is a subharmonic function. So from the maximum principle for subharmonic functions, and Theorem 5.2.11, for a fixed $w \in Y_\varepsilon$, we arrive at the estimate

$$\begin{aligned}\sup_{z \in U_\varepsilon(p)} \big|h_{\delta,p}(z,w)\big| &= \sup_{z \in \partial U_\varepsilon(p)} \big|h_{\delta,p}(z,w)\big| = \\ \sup_{z \in \partial U_\varepsilon(p)} &\left|g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,w)\right| \leq B_{\varepsilon,\alpha,\delta},\end{aligned}$$

for any $\alpha \in (0,\lambda_1)$ and $\delta > 0$. The proof of the corollary follows from the fact that the upper bound derived above does not depend on the fixed $w \in Y_\varepsilon$. □

Corollary 5.3.5. *Let $p, q \in \mathcal{P}$ be parabolic fixed points with $p \neq q$. Then, for any $\delta > 0$, $z \in U_\varepsilon(p)$, and $w \in U_\varepsilon(q)$, we have the relation*

$$g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) =$$
$$-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_q(w)|}{\log\varepsilon}\right) + h_{\delta,p,q}(z,w),$$

where $h_{\delta,p,q}(z,w)$ is a harmonic function in both the variables $z \in U_\varepsilon(p)$ and $w \in U_\varepsilon(q)$.

Proof. The proof of the corollary follows directly from arguments as in Lemma 5.3.3. □

Corollary 5.3.6. *Let $p, q \in \mathcal{P}$ be parabolic fixed points with $p \neq q$. Then, for any $\alpha \in (0, \lambda_1)$ and $\delta > 0$, we have the upper bound*

$$\sup_{\substack{z \in U_\varepsilon(p) \\ w \in U_\varepsilon(q)}} \big|h_{\delta,p,q}(z,w)\big| \leq B_{\varepsilon,\alpha,\delta}.$$

Proof. From Corollary 5.3.5, for any $\delta > 0$, $z \in U_\varepsilon(p)$, and $w \in U_\varepsilon(q)$, we have

$$g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) =$$
$$-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_q(w)|}{\log\varepsilon}\right) + h_{\delta,p,q}(z,w).$$

From the construction of the function $h_{\delta,p,q}(z,w)$, it follows that for any $\delta > 0$, $z \in \partial U_\varepsilon(p)$, and $w \in \partial U_\varepsilon(q)$, we find

$$h_{\delta,p,q}(z,w) = g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w).$$

As $h_{\delta,p,q}(z,w)$ is a harmonic function, $|h_{\delta,p,q}(z,w)|$ is a subharmonic function. So from the maximum principle for subharmonic functions, and Theorem 5.2.11, we arrive at the estimate

$$\sup_{\substack{z \in U_\varepsilon(p) \\ w \in U_\varepsilon(q)}} \big|h_{\delta,p,q}(z,w)\big| = \sup_{\substack{z \in \partial U_\varepsilon(p) \\ w \in \partial U_\varepsilon(q)}} \big|h_{\delta,p,q}(z,w)\big| =$$
$$\sup_{\substack{z \in \partial U_\varepsilon(p) \\ w \in \partial U_\varepsilon(q)}} \left|g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w)\right| \leq B_{\varepsilon,\alpha,\delta},$$

for any $\alpha \in (0, \lambda_1)$ and $\delta > 0$. This completes the proof of the corollary. □

Chapter 6

Bounds for canonical Green's functions

Recall that for $z, w \in X$, we have shown that

$$g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) = \phi(z) + \phi(w),$$

where

$$\phi(z) = \frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{8g^2}$$

with the constant C_{hyp} as defined in (3.36).

In this chapter, using the estimates of heat kernels and the hyperbolic Green's function obtained in Chapter 5, we derive bounds for the function $\phi(z)$. Using these bounds, we derive estimates of the canonical Green's function, after removing its log-singularity along the diagonal.

Recall that for a fixed $0 < \varepsilon < 1$ satisfying (5.1), we have defined

$$Y_\varepsilon = X \backslash \bigcup_{p \in \mathcal{P}} U_\varepsilon(p),$$

where $U_\varepsilon(p)$ is an open coordinate disk of radius ε around the parabolic fixed point $p \in \mathcal{P}$. From its definition, it follows that Y_ε is a compact subset of X.

In Section 6.1, we compute an estimate of the function $\phi(z)$ on the compact subset Y_ε of X. Using this estimate, we derive upper bounds for the canonical Green's function on the compact subset Y_ε of X, after removing its log-singularity along the diagonal.

In Section 6.2, we derive estimates of the function $\phi(z)$ in the neighborhoods of parabolic fixed points. Using these estimates, we derive upper bounds for the canonical Green's function in the neighborhoods of parabolic fixed points, after removing its log-singularity along the diagonal.

6.1 Bounds for the canonical Green's function on a compact subset

Recall that for $z \in X$, we have defined

$$P(z) = \sum_{\substack{\gamma \in \Gamma \setminus \{\mathrm{id}\} \\ \gamma \, \mathrm{parabolic}}} g_{\mathbb{H}}(z, \gamma z), \quad H(z) = 4\pi \int_0^\infty \left(HK_{\mathrm{hyp}}(t; z) - \frac{1}{\mathrm{vol_{hyp}}(X)} \right) dt$$

in Sections 4.2, 4.3, respectively. Furthermore, in Theorem 4.3.8, we have shown that

$$\phi(z) = \frac{H(z)}{2g} + \frac{1}{8\pi g} \int_X g_{\mathrm{hyp}}(z, \zeta) \, \Delta_{\mathrm{hyp}} \, P(\zeta) \, \mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{8g^2} - \frac{2\pi(c_X - 1)}{g \, \mathrm{vol_{hyp}}(X)},$$

where c_X is as defined in (4.1).

In this section, we obtain an upper bound for $\phi(z)$ on the compact subset Y_ε, by estimating each of the terms on the right-hand side of the above equation. We start by estimating the integral

$$\int_X g_{\mathrm{hyp}}(z, \zeta) \, \Delta_{\mathrm{hyp}} \, P(\zeta) \, \mu_{\mathrm{hyp}}(\zeta).$$

Using these bounds, we derive an upper bound for the function $\phi(z)$ on the compact subset Y_ε of X.

In the following lemma, we decompose the integral

$$\int_X g_{\mathrm{hyp}}(z, \zeta) \, \Delta_{\mathrm{hyp}} \, P(\zeta) \, \mu_{\mathrm{hyp}}(\zeta),$$

into five different terms.

Lemma 6.1.1. *For $z \in Y_\varepsilon$, we have the equality of integrals*

$$\int_X g_{\mathrm{hyp}}(z, \zeta) \, \Delta_{\mathrm{hyp}} \, P(\zeta) \, \mu_{\mathrm{hyp}}(\zeta) = 4\pi P(z) - 4\pi \int_{Y_{\varepsilon/2}} P(\zeta) \, \mu_{\mathrm{shyp}}(\zeta) +$$
$$4\pi \sum_{p \in \mathcal{P}} \left(\int_{\partial U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z, \zeta) d_\zeta^c P(\zeta) - \int_{\partial U_{\varepsilon/2}(p)} P(\zeta) d_\zeta^c g_{\mathrm{hyp}}(z, \zeta) \right) +$$
$$\sum_{p \in \mathcal{P}} \int_{U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z, \zeta) \, \Delta_{\mathrm{hyp}} \, P(\zeta) \, \mu_{\mathrm{hyp}}(\zeta).$$

Proof. From Lemma 4.2.8, we know that the function $\Delta_{\mathrm{hyp}} \, P(\zeta)$ is bounded on X. Furthermore, for a fixed $z \in Y_\varepsilon$, the hyperbolic Green's function $g_{\mathrm{hyp}}(z, \zeta)$ is log-singular at $\zeta = z$, and has $\log\log$-growth at the parabolic fixed points. So for any parabolic fixed point $p \in \mathcal{P}$, the integrals

$$\int_{Y_{\varepsilon/2}} g_{\mathrm{hyp}}(z, \zeta) \, \Delta_{\mathrm{hyp}} \, P(\zeta) \, \mu_{\mathrm{hyp}}(\zeta), \quad \int_{U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z, \zeta) \, \Delta_{\mathrm{hyp}} \, P(\zeta) \, \mu_{\mathrm{hyp}}(\zeta)$$

exist. Therefore, the decomposition of integrals is valid

$$\int_X g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta) =$$
$$\int_{Y_{\varepsilon/2}} g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta) + \sum_{p\in\mathcal{P}}\int_{U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta).$$

From the above equation, we deduce that, to prove the lemma, it suffices to prove that

$$\int_{Y_{\varepsilon/2}} g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta) = 4\pi P(z) - 4\pi\int_{Y_{\varepsilon/2}} P(\zeta)\,\mu_{\mathrm{shyp}}(\zeta)+$$
$$4\pi\sum_{p\in\mathcal{P}}\bigg(\int_{\partial U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)d_\zeta^c P(\zeta) - \int_{\partial U_{\varepsilon/2}(p)} P(\zeta)d_\zeta^c g_{\mathrm{hyp}}(z,\zeta)\bigg). \tag{6.1}$$

Let $U_r(z)$ denote an open coordinate disk of radius r around $z\in Y_{\varepsilon/2}$ with r small enough such that $U_r(z)\subsetneq Y_{\varepsilon/2}$. Using equation (6.1), we derive that, to prove the lemma, it suffices to show that

$$\int_{Y_{\varepsilon/2}\backslash U_r(z)} g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta) + 4\pi\int_{Y_{\varepsilon/2}\backslash U_r(z)} P(\zeta)\,\mu_{\mathrm{shyp}}(\zeta) \xrightarrow[r\to 0]{}$$
$$4\pi P(z) + 4\pi\sum_{p\in\mathcal{P}}\bigg(\int_{\partial U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)d_\zeta^c P(\zeta) - \int_{\partial U_{\varepsilon/2}(p)} P(\zeta)d_\zeta^c g_{\mathrm{hyp}}(z,\zeta)\bigg). \tag{6.2}$$

Observe that

$$\int_{Y_{\varepsilon/2}\backslash U_r(z)} g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta) + 4\pi\int_{Y_{\varepsilon/2}\backslash U_r(z)} P(\zeta)\,\mu_{\mathrm{shyp}}(\zeta) =$$
$$-4\pi\int_{Y_{\varepsilon/2}\backslash U_r(z)} g_{\mathrm{hyp}}(z,\zeta)d_\zeta d_\zeta^c P(\zeta) + 4\pi\int_{Y_{\varepsilon/2}\backslash U_r(z)} P(\zeta)d_\zeta d_\zeta^c g_{\mathrm{hyp}}(z,\zeta).$$

So combining the above equation with (6.2), it follows that, it suffices to show that

$$-4\pi\int_{Y_{\varepsilon/2}\backslash U_r(z)} g_{\mathrm{hyp}}(z,\zeta)d_\zeta d_\zeta^c P(\zeta) + 4\pi\int_{Y_{\varepsilon/2}\backslash U_r(z)} P(\zeta)d_\zeta d_\zeta^c g_{\mathrm{hyp}}(z,\zeta) \xrightarrow[r\to 0]{}$$
$$4\pi P(z) + 4\pi\sum_{p\in\mathcal{P}}\bigg(\int_{\partial U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)d_\zeta^c P(\zeta) - \int_{\partial U_{\varepsilon/2}(p)} P(\zeta)d_\zeta^c g_{\mathrm{hyp}}(z,\zeta)\bigg).$$

Using Stokes's theorem, we compute

$$-4\pi\int_{Y_{\varepsilon/2}\backslash U_r(z)} g_{\mathrm{hyp}}(z,\zeta)d_\zeta d_\zeta^c P(\zeta) + 4\pi\int_{Y_{\varepsilon/2}\backslash U_r(z)} P(\zeta)d_\zeta d_\zeta^c g_{\mathrm{hyp}}(z,\zeta) =$$
$$4\pi\int_{\partial U_r(z)} g_{\mathrm{hyp}}(z,\zeta)d_\zeta^c P(\zeta) - 4\pi\int_{\partial U_r(z)} P(\zeta)d_\zeta^c g_{\mathrm{hyp}}(z,\zeta)+$$
$$4\pi\sum_{p\in\mathcal{P}}\bigg(\int_{\partial U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)d_\zeta^c P(\zeta) - \int_{\partial U_{\varepsilon/2}(p)} P(\zeta)d_\zeta^c g_{\mathrm{hyp}}(z,\zeta)\bigg). \tag{6.3}$$

As $Y_{\varepsilon/2}$ is compact and $P(\zeta)$ is smooth on $Y_{\varepsilon/2}$, using equation (2.14) from the proof of Lemma 2.5.4, we deduce that

$$4\pi\int_{\partial U_r(z)} g_{\mathrm{hyp}}(z,\zeta)d^c_\zeta P(\zeta) - 4\pi\int_{\partial U_r(z)} P(\zeta)d^c_\zeta g_{\mathrm{hyp}}(z,\zeta) \xrightarrow[r\to 0]{} 4\pi P(z). \quad (6.4)$$

The proof of the lemma follows from (6.3) by letting r approach zero in combination with (6.4). □

Corollary 6.1.2. *For $z\in Y_\varepsilon$, we have*

$$\phi(z) = \frac{1}{2g}\bigg(H(z)+P(z)-\int_{Y_{\varepsilon/2}} P(\zeta)\,\mu_{\mathrm{shyp}}(\zeta)\bigg)+$$
$$\frac{1}{2g}\sum_{p\in\mathcal{P}}\bigg(\int_{\partial U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)d^c_\zeta P(\zeta) - \int_{\partial U_{\varepsilon/2}(p)} P(\zeta)d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\bigg)+$$
$$\frac{1}{8\pi g}\sum_{p\in\mathcal{P}}\int_{U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{8g^2} - \frac{2\pi(c_X-1)}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)}, \quad (6.5)$$

where C_{hyp} and c_X are as defined in (3.36) and (4.1), respectively.

Proof. The corollary follows by combining Theorem 4.3.8 with Lemma 6.1.1. □

We will now obtain bounds for each of the quantities on the right-hand side of equation (6.5).

Remark 6.1.3. As the bounds for each of the quantities on the right-hand side of equation (6.5) involve the terms λ_1, r_ε, c_ε, d_X, and $B_{\varepsilon,\alpha,\delta}$, we recall their definitions again. Here, λ_1 is the first non-zero eigenvalue of the hyperbolic Laplacian Δ_{hyp}, the injectivity radius r_ε is as defined in (5.2), the constant c_ε is as defined in (5.15), the constant d_X is as given by (1.5), and the constant $B_{\varepsilon,\alpha,\delta}$ is as defined in Theorem 5.2.11.
Furthermore, C'_{par} and C''_{par} are as defined in (4.19) and (4.21), respectively.

Using an estimate of the hyperbolic Green's function that we derived in Chapter 5, the first two terms on the right-hand side of equation (6.5) can easily be estimated, which is accomplished in the following proposition.

Proposition 6.1.4. *With the recollections of Remark 6.1.3, for any $\alpha\in(0,\lambda_1)$ and $\delta\in(0,r_{\varepsilon/2})$, we have the upper bound*

$$\sup_{z\in Y_\varepsilon}|H(z)+P(z)|\leq B_{\varepsilon/2,\alpha,\delta}.$$

Proof. From equation (4.23), we know that

$$H(z)+P(z) = \lim_{w\to z}\big(g_{\mathrm{hyp}}(z,w)-g_{\mathbb{H}}(z,w)\big).$$

Using the above equation, for any $\alpha \in (0, \lambda_1)$ and $\delta \in (0, r_{\varepsilon/2})$, we have

$$\sup_{z \in Y_\varepsilon} |H(z) + P(z)| = \sup_{z \in Y_\varepsilon} \left| \lim_{w \to z} \big(g_{\mathrm{hyp}}(z,w) - g_{\mathbb{H}}(z,w) \big) \right| \leq$$
$$\sup_{z \in Y_{\varepsilon/2}} \left| \lim_{w \to z} \big(g_{\mathrm{hyp}}(z,w) - g_{\mathbb{H}}(z,w) \big) \right| =$$
$$\sup_{z \in Y_{\varepsilon/2}} \left| \lim_{w \to z} \Bigg(g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z, w)} g_{\mathbb{H}}(z, \gamma w) \Bigg) - \sum_{n\colon 0 < \lambda_n < \alpha} \frac{4\pi}{\lambda_n} \varphi_n^2(z) \right|.$$

Now the claimed estimate follows from Theorem 5.2.11. $\square$

In the following proposition, we derive a bound for the third term on the right-hand side of equation (6.5).

Proposition 6.1.5. *We have the upper bound*

$$\left| \int_{Y_{\varepsilon/2}} P(z)\, \mu_{\mathrm{shyp}}(z) \right| \leq -2\, |\mathcal{P}| \log(\varepsilon/2),$$

where $|\mathcal{P}|$ *denotes the number of parabolic fixed points of* Γ.

Proof. Since $P(z)$ is a non-negative function on X, from equation (4.2), we have the estimate

$$\int_{Y_{\varepsilon/2}} P(z)\, \mu_{\mathrm{shyp}}(z) = \int_{Y'_{\varepsilon/2}} \sum_{p \in \mathcal{P}} \sum_{\eta \in \Gamma_p \backslash \Gamma} P_{\mathrm{gen},p}(\eta z)\, \mu_{\mathrm{shyp}}(z) \geq 0,$$

where $Y'_{\varepsilon/2}$ is as defined in Notation 5.1.2 and

$$P_{\mathrm{gen},p}(z) = \sum_{n \neq 0} g_{\mathbb{H}}(z, \gamma_p^n z),$$

where γ_p is a generator of the stabilizer subgroup Γ_p of the parabolic fixed point $p \in \mathcal{P}$. Interchanging summation and integration, we obtain

$$\int_{Y'_{\varepsilon/2}} \sum_{p \in \mathcal{P}} \sum_{\eta \in \Gamma_p \backslash \Gamma} P_{\mathrm{gen},p}(\eta z)\, \mu_{\mathrm{shyp}}(z) = \sum_{p \in \mathcal{P}} \sum_{\eta \in \Gamma_p \backslash \Gamma} \int_{Y'_{\varepsilon/2}} P_{\mathrm{gen},p}(\eta z)\, \mu_{\mathrm{shyp}}(z). \tag{6.6}$$

The interchange of summation and integration in the above equation is valid, provided that the latter series converges absolutely, which we prove now. For $p \in \mathcal{P}$ a parabolic fixed point, recall that

$$Y'_{\varepsilon/2,p} = Y'_{\varepsilon/2} \cup \bigcup_{\substack{q \in \mathcal{P} \\ q \neq p}} U'_{\varepsilon/2}(q).$$

Since the function $P_{\mathrm{gen},p}(\eta z)$ is non-negative on $\mathbb{H}$, we have

$$\sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma}\int_{Y'_{\varepsilon/2}} P_{\mathrm{gen},p}(\eta z)\,\mu_{\mathrm{shyp}}(z) \leq \sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma}\int_{Y'_{\varepsilon/2,p}} P_{\mathrm{gen},p}(\eta z)\,\mu_{\mathrm{shyp}}(z).$$

After making the substitution $z\mapsto \eta^{-1}\sigma_p z$, from the $\mathrm{PSL}_2(\mathbb{R})$-invariance of the metric $\mu_{\mathrm{shyp}}(z)$, we get

$$\sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma}\int_{Y'_{\varepsilon/2,p}} P_{\mathrm{gen},p}(\eta z)\,\mu_{\mathrm{shyp}}(z) =$$
$$\sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma}\int_{\sigma_p^{-1}\eta Y'_{\varepsilon/2,p}} P_{\mathrm{gen},p}(\sigma_p z)\,\mu_{\mathrm{shyp}}(z).$$

From Remark 5.1.3, we have

$$\sum_{p\in\mathcal{P}}\sum_{\eta\in\Gamma_p\backslash\Gamma}\int_{\sigma_p^{-1}\eta Y'_{\varepsilon/2,p}} P_{\mathrm{gen},p}(\sigma_p z)\,\mu_{\mathrm{shyp}}(z) =$$
$$\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\sum_{p\in\mathcal{P}}\int_0^{-\frac{\log(\varepsilon/2)}{2\pi}}\int_0^1 P_{\mathrm{gen},p}(\sigma_p z)\frac{dxdy}{y^2}.$$

From the estimate obtained in equation (4.9), we derive

$$\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\sum_{p\in\mathcal{P}}\int_0^{-\frac{\log(\varepsilon/2)}{2\pi}}\int_0^1 P_{\mathrm{gen},p}(\sigma_p z)\frac{dxdy}{y^2} \leq$$
$$\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\sum_{p\in\mathcal{P}}\int_0^{-\frac{\log(\varepsilon/2)}{2\pi}}\int_0^1 32y^2\frac{dxdy}{y^2} = -\frac{16\,|\mathcal{P}|\log(\varepsilon/2)}{\pi\,\mathrm{vol}_{\mathrm{hyp}}(X)}.$$

Using the fact that $2\pi\leq \mathrm{vol}_{\mathrm{hyp}}(X)$, we arrive at

$$\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\sum_{p\in\mathcal{P}}\int_0^{-\frac{\log(\varepsilon/2)}{2\pi}}\int_0^1 P_{\mathrm{gen},p}(\sigma_p z)\frac{dxdy}{y^2} \leq$$
$$-\frac{16\,|\mathcal{P}|\log(\varepsilon/2)}{\pi\,\mathrm{vol}_{\mathrm{hyp}}(X)} \leq -\frac{16\,|\mathcal{P}|\log(\varepsilon/2)}{2\pi^2} \leq -2\,|\mathcal{P}|\log(\varepsilon/2).$$

This proves that the right-hand side of equation (6.6) converges absolutely. Hence, the interchange of summation and integration in equation (6.6) remains valid. The claim of the proposition now follows from the above estimate. $\square$

In the following proposition, we derive a bound for the fourth term on the right-hand side of equation (6.5).

Proposition 6.1.6. *With the recollections of Remark 6.1.3, for any $\alpha\in(0,\lambda_1)$ and $\delta\in(0,c_\varepsilon)$, we have the upper bound*

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\left|\int_{\partial U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)d^c_\zeta P(\zeta)\right| \leq |\mathcal{P}|\,B_{\varepsilon/2,\alpha,\delta}.$$

Proof. Observe the elementary estimate

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\bigg|\int_{\partial U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)d^c_\zeta P(\zeta)\bigg|\leq$$

$$\sup_{\substack{z\in Y_\varepsilon\\ \zeta\in\partial Y_{\varepsilon/2}}}\big|g_{\mathrm{hyp}}(z,\zeta)\big|\cdot\bigg(\sum_{p\in\mathcal{P}}\bigg|\int_{\partial U_{\varepsilon/2}(p)} d^c_\zeta P(\zeta)\bigg|\bigg). \tag{6.7}$$

From Corollary 5.2.13, we have the upper bound

$$\sup_{\substack{z\in Y_\varepsilon\\ \zeta\in\partial Y_{\varepsilon/2}}}|g_{\mathrm{hyp}}(z,\zeta)|\leq B_{\varepsilon/2,\alpha,\delta}, \tag{6.8}$$

for any $\alpha\in(0,\lambda_1)$ and $\delta\in(0,c_\varepsilon)$. We are now left to estimate the sum

$$\sum_{p\in\mathcal{P}}\bigg|\int_{\partial U_{\varepsilon/2}(p)} d^c_\zeta P(\zeta)\bigg|.$$

From Stokes's theorem, we have

$$\sum_{p\in\mathcal{P}}\bigg|\int_{\partial U_{\varepsilon/2}(p)} d^c_\zeta P(\zeta)\bigg| = \frac{1}{4\pi}\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)}\Delta_{\mathrm{hyp}}P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg|.$$

From Remark 4.2.10, we know that $\Delta_{\mathrm{hyp}}P(\zeta)$ is a non-positive function on X. So we arrive at the estimate

$$\frac{1}{4\pi}\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)}\Delta_{\mathrm{hyp}}P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg| =$$

$$\frac{1}{4\pi}\sum_{p\in\mathcal{P}}\int_{U_{\varepsilon/2}(p)}|\,\Delta_{\mathrm{hyp}}P(\zeta)|\,\mu_{\mathrm{hyp}}(\zeta)\leq\frac{1}{4\pi}\int_X|\,\Delta_{\mathrm{hyp}}P(\zeta)|\,\mu_{\mathrm{hyp}}(\zeta).$$

We now try to estimate the integral

$$\int_X\Delta_{\mathrm{hyp}}P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta).$$

From Corollary 3.2.5, we have

$$\int_X g\,\mu_{\mathrm{can}}(\zeta)=$$

$$\bigg(\frac{1}{4\pi}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)\int_X\mu_{\mathrm{hyp}}(\zeta)+\frac{1}{2}\int_X\bigg(\int_0^\infty\Delta_{\mathrm{hyp}}K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\mu_{\mathrm{hyp}}(\zeta).$$

Using the above equation and the fact that

$$\mathrm{vol}_{\mathrm{hyp}}(X)=2\pi\big(2g-2+|\mathcal{P}|\big),$$

we compute

$$\int_X \left(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;\zeta)dt \right) \mu_{\mathrm{hyp}}(\zeta) =$$
$$2g - 2\,\mathrm{vol}_{\mathrm{hyp}}(X) \left(\frac{1}{4\pi} + \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) = -|\mathcal{P}|.$$

Furthermore, using Lemma 4.3.7, we derive

$$4\pi \int_X \left(\int_0^\infty \Delta_{\mathrm{hyp}} K_{\mathrm{hyp}}(t;\zeta)dt \right) \mu_{\mathrm{hyp}}(\zeta) =$$
$$\int_X \Delta_{\mathrm{hyp}} H(\zeta)\, \mu_{\mathrm{hyp}}(\zeta) + \int_X \Delta_{\mathrm{hyp}} P(\zeta)\, \mu_{\mathrm{hyp}}(\zeta) = -4\pi\, |\mathcal{P}|.$$

For the subsequent calculation, we let $U_r(p)$ denote an open coordinate disk of radius r around a parabolic fixed point $p \in \mathcal{P}$ and set

$$Y_r = X \setminus \bigcup_{p \in \mathcal{P}} U_r(p).$$

Then, using Stokes's theorem, we find

$$\int_X \Delta_{\mathrm{hyp}} H(\zeta)\, \mu_{\mathrm{hyp}}(\zeta) = -4\pi \lim_{r \to 0} \int_{Y_r} d_\zeta d_\zeta^c H(\zeta) =$$
$$4\pi \sum_{p \in \mathcal{P}} \lim_{r \to 0} \int_{\partial U_r(p)} d_\zeta^c H(\zeta) = 4\pi \sum_{p \in \mathcal{P}} \lim_{r \to 0} \int_0^{2\pi} \frac{r}{2} \frac{\partial H(\zeta)}{\partial r} \frac{d\theta}{2\pi}.$$

Using Corollary 4.3.4, for $p \in \mathcal{P}$ and $\zeta \in \partial U_r(p)$, we compute

$$\begin{aligned} \lim_{r \to 0} \int_0^{2\pi} \frac{r}{2} \frac{\partial H(\zeta)}{\partial r} \frac{d\theta}{2\pi} &= -\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \lim_{r \to 0} \left(\int_0^{2\pi} \frac{r}{2} \frac{\partial \log\big(-\log r\big)}{\partial r} \frac{d\theta}{2\pi} + O(r) \right) \\ &= -\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \lim_{r \to 0} \frac{1}{2\log r} = 0. \end{aligned}$$

This implies that

$$\int_X \Delta_{\mathrm{hyp}} H(\zeta)\, \mu_{\mathrm{hyp}}(\zeta) = 0 \;\text{ and }\; \int_X |\,\Delta_{\mathrm{hyp}} P(\zeta)|\, \mu_{\mathrm{hyp}}(\zeta) = 4\pi\, |\mathcal{P}|.$$

So we arrive at

$$\sum_{p \in \mathcal{P}} \left| \int_{\partial U_{\varepsilon/2}(p)} d_\zeta^c P(\zeta) \right| = \frac{1}{4\pi} \sum_{p \in \mathcal{P}} \left| \int_{U_{\varepsilon/2}(p)} \Delta_{\mathrm{hyp}} P(\zeta)\, \mu_{\mathrm{hyp}}(\zeta) \right| \leq |\mathcal{P}|. \tag{6.9}$$

Hence, combining the estimates obtained in (6.7), (6.8), and (6.9), for any $\alpha \in (0, \lambda_1)$ and $\delta \in (0, c_\varepsilon)$, we derive

$$\sup_{z \in Y_\varepsilon} \sum_{p \in \mathcal{P}} \left| \int_{\partial U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta) d_\zeta^c P(\zeta) \right| \leq |\mathcal{P}|\, B_{\varepsilon/2,\alpha,\delta}.$$

This completes the proof of the proposition. □

In the following proposition, we derive a bound for the fifth term on the right-hand side of equation (6.5).

Proposition 6.1.7. *With the recollections of Remark 6.1.3, we have the upper bound*

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\left|\int_{\partial U_{\varepsilon/2}(p)} P(\zeta)d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\right| \leq -2\,|\mathcal{P}|\log(\varepsilon/2)+C'_{\mathrm{par}}.$$

Proof. Since $P(\zeta)$ is a non-negative function on X, we have

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\left|\int_{\partial U_{\varepsilon/2}(p)} P(\zeta)d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\right| \leq$$
$$\sup_{\zeta\in Y_{\varepsilon/2}} P(\zeta)\cdot\left(\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\left|\int_{\partial U_{\varepsilon/2}(p)} d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\right|\right). \tag{6.10}$$

We now bound the product on the right-hand side of the above inequality, term by term. Using Stokes's theorem, we have the upper bound for the second term

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\left|\int_{\partial U_{\varepsilon/2}(p)} d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\right| =$$
$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\left|\int_{U_{\varepsilon/2}(p)} d_\zeta d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\right| = \sum_{p\in\mathcal{P}}\int_{U_{\varepsilon/2}(p)}\mu_{\mathrm{shyp}}(\zeta)\leq 1. \tag{6.11}$$

We now compute an upper bound for the term

$$\sup_{\zeta\in Y_{\varepsilon/2}} P(\zeta).$$

Using the decomposition stated in equation (4.2), we find

$$\sup_{\zeta\in Y_{\varepsilon/2}} P(\zeta)\leq \sup_{\zeta\in Y'_{\varepsilon/2}}\sum_{p\in\mathcal{P}}\sum_{\substack{\eta\in\Gamma_p\backslash\Gamma\\ \eta\neq\mathrm{id}}} P_{\mathrm{gen},p}(\eta\zeta)+\sup_{\zeta\in Y'_{\varepsilon/2}}\sum_{p\in\mathcal{P}} P_{\mathrm{gen},p}(\zeta). \tag{6.12}$$

We now try to estimate the right-hand side of the above inequality term by term. Recalling the definition of the constant C'_{par} from (4.19), we have the upper bound for the first term

$$\sup_{\zeta\in Y'_{\varepsilon/2}}\sum_{p\in\mathcal{P}}\sum_{\substack{\eta\in\Gamma_p\backslash\Gamma\\ \eta\neq\mathrm{id}}} P_{\mathrm{gen},p}(\eta\zeta)\leq C'_{\mathrm{par}}. \tag{6.13}$$

For the second term on the right-hand side of equation (6.12), using the estimate computed in equation (4.9), we derive the inequality

$$\sup_{\zeta\in Y'_{\varepsilon/2}}\sum_{p\in\mathcal{P}} P_{\mathrm{gen},p}(\zeta)\leq$$
$$\sup_{\zeta\in Y'_{\varepsilon/2}}\sum_{p\in\mathcal{P}}\left(4\pi\,\mathrm{Im}(\sigma_p^{-1}\zeta)-8\,\mathrm{Im}(\sigma_p^{-1}\zeta)\tan^{-1}\left(\frac{1}{2\,\mathrm{Im}(\sigma_p^{-1}\zeta)}\right)\right)\leq$$
$$\sup_{\zeta\in Y'_{\varepsilon/2}}\sum_{p\in\mathcal{P}} 4\pi\,\mathrm{Im}(\sigma_p^{-1}\zeta) = -2\,|\mathcal{P}|\log(\varepsilon/2).$$

Combining the above estimate with (6.13), we get

$$\sup_{\zeta\in Y_{\varepsilon/2}} P(\zeta) \leq -2\,|\mathcal{P}|\log(\varepsilon/2) + C'_{\mathrm{par}}. \tag{6.14}$$

Hence, combining the estimates obtained in (6.10), (6.11), and (6.14), we arrive at the upper bound

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\bigg|\int_{\partial U_\varepsilon(p)} P(\zeta)d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\bigg| \leq -2\,|\mathcal{P}|\log(\varepsilon/2) + C'_{\mathrm{par}}.$$

This completes the proof of the proposition. □

In the following proposition, we derive a bound for the sixth term on the right-hand side of equation (6.5).

Proposition 6.1.8. *With the recollections of Remark 6.1.3, for any $\alpha\in(0,\lambda_1)$ and $\delta\in(0,c_\varepsilon)$, we have the upper bound*

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg| \leq$$
$$-\frac{2\pi\,|\mathcal{P}|\,C''_{\mathrm{par}}}{\log(\varepsilon/2)}\bigg(B_{\varepsilon/2,\alpha,\delta}+\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg).$$

Proof. Observe the inequality

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg| \leq$$
$$\sup_{\zeta\in X}\big|\,\Delta_{\mathrm{hyp}}\,P(\zeta)\big|\cdot\bigg(\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg|\bigg).$$

We now estimate the product on the right-hand side of the above inequality term by term.

Recalling the definition of the constant C''_{par} from (4.21), we have an upper bound for the first term in the product on the right-hand side of the above inequality

$$\sup_{\zeta\in X}\big|\,\Delta_{\mathrm{hyp}}\,P(\zeta)\big| \leq C''_{\mathrm{par}}. \tag{6.15}$$

We are left to estimate the quantity

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg|.$$

From Lemma 5.3.1, we have

$$\sum_{p\in\mathcal{P}}\int_{U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta) = \sum_{p\in\mathcal{P}}\int_{U_{\varepsilon/2}(p)} g_p(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta) -$$
$$\sum_{p\in\mathcal{P}}\int_{U_{\varepsilon/2}(p)}\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\bigg(\frac{\log|\vartheta_p(\zeta)|}{\log(\varepsilon/2)}\bigg)\,\mu_{\mathrm{hyp}}(\zeta), \tag{6.16}$$

where $g_p(z,\zeta)$ is a harmonic function in the variable $\zeta \in U_{\varepsilon/2}(p)$ for each $p \in \mathcal{P}$.

We now derive an upper bound for the quantity on the right-hand side of the above equation, term by term. Using the estimate derived in Corollary 5.3.2, we compute

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)} g_p(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg| \leq$$
$$\sum_{p\in\mathcal{P}}\sup_{\substack{z\in Y_\varepsilon\\ \zeta\in U_{\varepsilon/2}(p)}}\big|g_p(z,\zeta)\big|\cdot\int_{U_{\varepsilon/2}(p)}\mu_{\mathrm{hyp}}(\zeta) \leq B_{\varepsilon/2,\alpha,\delta}\cdot\bigg(\sum_{p\in\mathcal{P}}\int_{U_{\varepsilon/2}(p)}\mu_{\mathrm{hyp}}(\zeta)\bigg),$$

for any $\alpha \in (0,\lambda_1)$ and $\delta \in (0,c_\varepsilon)$. For any parabolic fixed point $p \in \mathcal{P}$, we make the volume computation

$$\int_{U_{\varepsilon/2}(p)}\mu_{\mathrm{hyp}}(\zeta) = \int_0^{\varepsilon/2}\int_0^{2\pi}\frac{rdrd\theta}{(r\log r)^2} = 2\pi\int_0^{\varepsilon/2}\frac{d(\log r)}{(\log r)^2} = -\frac{2\pi}{\log(\varepsilon/2)}. \tag{6.17}$$

Using the above computation, we arrive at the upper bound for the first term on the right-hand side of equation (6.16)

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)} g_p(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg| \leq -\frac{2\pi\,|\mathcal{P}|\,B_{\varepsilon/2,\alpha,\delta}}{\log(\varepsilon/2)}, \tag{6.18}$$

for any $\alpha \in (0,\lambda_1)$ and $\delta \in (0,c_\varepsilon)$.

We now compute the integral

$$\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)}\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\bigg(\frac{\log|\vartheta_p(\zeta)|}{\log(\varepsilon/2)}\bigg)\bigg|\,\mu_{\mathrm{hyp}}(\zeta),$$

which is the second term on the right-hand side of equation (6.16). From equation (3.3), we have

$$\begin{aligned}\int_{U_{\varepsilon/2}(p)}\log\big(-\log|\vartheta_p(\zeta)|\big)\,\mu_{\mathrm{hyp}}(\zeta) \;&=\; \int_0^{\varepsilon/2}\int_0^{2\pi}\frac{r\log\big(-\log r\big)drd\theta}{(r\log r)^2}\\ &=\; -\frac{2\pi}{\log(\varepsilon/2)}\bigg(\log\big(-\log(\varepsilon/2)\big)+1\bigg).\end{aligned} \tag{6.19}$$

Using equations (6.17) and (6.19), we derive

$$\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)}\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\bigg(\frac{\log|\vartheta_p(\zeta)|}{\log(\varepsilon/2)}\bigg)\bigg|\,\mu_{\mathrm{hyp}}(\zeta) =$$
$$\sum_{p\in\mathcal{P}}\int_{U_{\varepsilon/2}(p)}\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\bigg(\frac{-\log|\vartheta_p(\zeta)|}{-\log(\varepsilon/2)}\bigg)\,\mu_{\mathrm{hyp}}(\zeta) = -\frac{8\pi^2\,|\mathcal{P}|}{\mathrm{vol}_{\mathrm{hyp}}(X)\log(\varepsilon/2)}. \tag{6.20}$$

Hence, combining the estimate derived in (6.18) and equation (6.20), we arrive at the upper bound

$$\sup_{z\in Y_{\varepsilon}}\sum_{p\in\mathcal{P}}\bigg|\int_{U_{\varepsilon/2}(p)}g_{\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg|\leq-\frac{2\pi\,|\mathcal{P}|}{\log(\varepsilon/2)}\bigg(B_{\varepsilon/2,\alpha,\delta}+\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg),\tag{6.21}$$

for any $\alpha\in(0,\lambda_1)$ and $\delta\in(0,c_{\varepsilon})$. Finally, combining the estimates obtained in (6.15) and (6.21), we get

$$\sup_{z\in Y_{\varepsilon}}\bigg|\int_{U_{\varepsilon/2}(p)}g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg|\leq$$
$$-\frac{2\pi\,|\mathcal{P}|\,C''_{\mathrm{par}}}{\log(\varepsilon/2)}\bigg(B_{\varepsilon/2,\alpha,\delta}+\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg),$$

for any $\alpha\in(0,\lambda_1)$ and $\delta\in(0,c_{\varepsilon})$. This completes the proof of the proposition. □

In the following proposition, we derive a bound for the double integral C_{hyp}, using techniques similar to the ones used to prove Proposition 4.1 in [12].

Proposition 6.1.9. *With the recollections of Remark 6.1.3, we have the upper bound*

$$C_{\mathrm{hyp}}\leq\frac{16\pi g^2\,(d_X+1)^2}{\lambda_1\,\mathrm{vol}_{\mathrm{hyp}}(X)}.$$

Proof. In order to prove the proposition, it will be useful to prove an alternative formula for C_{hyp}. To derive this formula, we recall that C_{hyp} is defined as (see (3.36))

$$C_{\mathrm{hyp}}=\int_X\int_X g_{\mathrm{hyp}}(\zeta,\xi)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\times$$
$$\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\xi)dt\bigg)\,\mu_{\mathrm{hyp}}(\xi)\,\mu_{\mathrm{hyp}}(\zeta).$$

From equation (4.36), we have

$$\phi(z)=\frac{1}{2g}\int_X g_{\mathrm{hyp}}(z,\zeta)\bigg(\int_0^{\infty}\Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\,\mu_{\mathrm{hyp}}(\zeta)-\frac{C_{\mathrm{hyp}}}{8g^2}.\tag{6.22}$$

Furthermore, from Proposition 2.6.4, we have

$$d_zd_z^c\phi(z)=\mu_{\mathrm{shyp}}(z)-\mu_{\mathrm{can}}(z),\tag{6.23}$$

from which we derive

$$\int_X d_zd_z^c\phi(z)=0.$$

So combining equations (6.22) and (6.23), we get

$$
\begin{aligned}
&-\frac{1}{4\pi}\int_X \phi(z)\,\Delta_{\mathrm{hyp}}\,\phi(z)\,\mu_{\mathrm{hyp}}(z)=\int_X \phi(z)d_zd_z^c\phi(z)=\\
&\frac{1}{2g}\int_X\int_X g_{\mathrm{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\,\mu_{\mathrm{hyp}}(\zeta)\,\mu_{\mathrm{shyp}}(z)-\\
&\frac{1}{2g}\int_X\int_X g_{\mathrm{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\,\mu_{\mathrm{hyp}}(\zeta)\,\mu_{\mathrm{can}}(z)-\\
&\frac{C_{\mathrm{hyp}}}{8g^2}\int_X d_zd_z^c\phi(z).
\end{aligned}
$$

Observe that the first and third integrals on the right-hand side of the above equation are zero. Therefore, we arrive at

$$
\begin{aligned}
&\int_X \phi(z)\,\Delta_{\mathrm{hyp}}\,\phi(z)\,\mu_{\mathrm{hyp}}(z)=\\
&\frac{2\pi}{g}\int_X\int_X g_{\mathrm{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\,\mu_{\mathrm{hyp}}(\zeta)\,\mu_{\mathrm{can}}(z).
\end{aligned}
\tag{6.24}
$$

From Theorem 1.11.2, we have the expression for the canonical metric $\mu_{\mathrm{can}}(z)$

$$
\begin{aligned}
&g\,\mu_{\mathrm{can}}(z)=\\
&\bigg(\frac{1}{4\pi}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)\,\mu_{\mathrm{hyp}}(z)+\frac{1}{2}\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;z)dt\bigg)\,\mu_{\mathrm{hyp}}(z).
\end{aligned}
$$

Substituting the expression on the right-hand side of the above equation for the canonical metric $\mu_{\mathrm{can}}(z)$ in equation (6.24), we get

$$
\begin{aligned}
&\int_X \phi(z)\,\Delta_{\mathrm{hyp}}\,\phi(z)\,\mu_{\mathrm{hyp}}(z)=\frac{\pi}{g^2}\int_X\int_X g_{\mathrm{hyp}}(z,\zeta)\times\\
&\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;\zeta)dt\bigg)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{\mathrm{hyp}}(t;z)dt\bigg)\,\mu_{\mathrm{hyp}}(\zeta)\,\mu_{\mathrm{hyp}}(z)=\frac{\pi}{g^2}C_{\mathrm{hyp}},
\end{aligned}
$$

which implies that

$$
C_{\mathrm{hyp}}=\frac{g^2}{\pi}\int_X \phi(z)\,\Delta_{\mathrm{hyp}}\,\phi(z)\,\mu_{\mathrm{hyp}}(z). \tag{6.25}
$$

Using the above formula, we now derive estimates for the constant C_{hyp}. From equation (6.23), we have

$$
\Delta_{\mathrm{hyp}}\,\phi(z)=\frac{4\pi\,\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)}-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}.
$$

Using the above relation, we compute the upper bound

$$
\sup_{z\in X}|\,\Delta_{\mathrm{hyp}}\,\phi(z)|\leq\sup_{z\in X}\bigg|\frac{4\pi\,\mu_{\mathrm{can}}(z)}{\mathrm{vol}_{\mathrm{hyp}}(X)\,\mu_{\mathrm{shyp}}(z)}\bigg|+\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}=\frac{4\pi\,(d_X+1)}{\mathrm{vol}_{\mathrm{hyp}}(X)}, \tag{6.26}
$$

where d_X is as defined in (1.5). From Remark 2.6.7, we have the spectral expansion for the function $\phi(z)$

$$\phi(z) = \sum_{n=0}^{\infty} \phi_n \varphi_n(z) + \frac{1}{4\pi} \sum_{p\in\mathcal{P}} \int_{-\infty}^{\infty} \phi_p(r) \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) dr, \tag{6.27}$$

where $\phi_n = \big\langle \phi(z), \varphi_n(z) \big\rangle$ and $\phi_p(r) = \big\langle \phi(z), \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) \big\rangle$.

Applying the Laplace operator to both sides of the above equation, we find

$$\Delta_{\mathrm{hyp}} \phi(z) =$$
$$\sum_{n=1}^{\infty} \lambda_n \phi_n \varphi_n(z) + \frac{1}{4\pi} \sum_{p\in\mathcal{P}} \int_{-\infty}^{\infty} \left(r^2 + \frac{1}{4}\right) \phi_p(r) \mathcal{E}_{\mathrm{par},p}(z, 1/2 + ir) dr. \tag{6.28}$$

Using the above equation and Proposition 1.6.4, we deduce that

$$\int_X |\Delta_{\mathrm{hyp}} \phi(z)|^2 \mu_{\mathrm{hyp}}(z) = \sum_{n=1}^{\infty} \lambda_n^2 \big|\phi_n\big|^2 + \frac{1}{4\pi} \sum_{p\in\mathcal{P}} \int_{-\infty}^{\infty} \left(r^2 + \frac{1}{4}\right)^2 \big|\phi_p(r)\big|^2 dr. \tag{6.29}$$

Recall from Remark 5.2.7 that we have chosen the eigenfunctions $\{\varphi_n(z)\}$ of the hyperbolic Laplacian Δ_{hyp} to be real-valued. Furthermore, both the functions $\phi(z)$ and $\Delta_{\mathrm{hyp}} \phi(z)$ are real-valued. So using equations (6.27) and (6.28), and Proposition 1.6.4, we compute

$$\int_X \phi(z) \Delta_{\mathrm{hyp}} \phi(z) \mu_{\mathrm{hyp}}(z) = \sum_{n=1}^{\infty} \lambda_n \big|\phi_n\big|^2 + \frac{1}{4\pi} \sum_{p\in\mathcal{P}} \int_{-\infty}^{\infty} \left(r^2 + \frac{1}{4}\right) \big|\phi_p(r)\big|^2 dr. \tag{6.30}$$

Using the fact that $\lambda_n \geq \lambda_1$ for all $n \in \mathbb{N}_{\geq 1}$ and $0 < \lambda_1 \leq 1/4$, we arrive at the two inequalities

$$\sum_{n=1}^{\infty} \lambda_1 \lambda_n \big|\phi_n\big|^2 \leq \sum_{n=1}^{\infty} \lambda_n^2 \big|\phi_n\big|^2,$$
$$\sum_{p\in\mathcal{P}} \lambda_1 \int_{-\infty}^{\infty} \left(r^2 + \frac{1}{4}\right) \big|\phi_p(r)\big|^2 dr \leq \sum_{p\in\mathcal{P}} \int_{-\infty}^{\infty} \left(r^2 + \frac{1}{4}\right)^2 \big|\phi_p(r)\big|^2 dr.$$

From the above two inequalities, and equations (6.29) and (6.30), it then follows that

$$\int_X \phi(z) \Delta_{\mathrm{hyp}} \phi(z) \mu_{\mathrm{hyp}}(z) \leq \frac{1}{\lambda_1} \int_X |\Delta_{\mathrm{hyp}} \phi(z)|^2 \mu_{\mathrm{hyp}}(z).$$

Using the above estimate, and the upper bound derived in (6.26), we deduce that

$$\int_X \phi(z) \Delta_{\mathrm{hyp}} \phi(z) \mu_{\mathrm{hyp}}(z) \leq \frac{1}{\lambda_1} \int_X |\Delta_{\mathrm{hyp}} \phi(z)|^2 \mu_{\mathrm{hyp}}(z) \leq \frac{16\pi^2 \left(d_X + 1\right)^2}{\lambda_1 \operatorname{vol}_{\mathrm{hyp}}(X)}.$$

Hence, from equation (6.25), and the above estimate, we arrive at the upper bound

$$C_{\mathrm{hyp}} = \frac{g^2}{\pi}\int_X \phi(z)\,\Delta_{\mathrm{hyp}}\,\phi(z)\,\mu_{\mathrm{hyp}}(z) \leq \frac{16\pi g^2\,(d_X+1)^2}{\lambda_1\,\mathrm{vol}_{\mathrm{hyp}}(X)}.$$

This completes the proof of the proposition. □

Definition 6.1.10. Put

$$\widetilde{r}_\varepsilon = \min\{r_{\varepsilon/2}, c_\varepsilon\} \tag{6.31}$$

Collecting all the estimates that we have derived so far in this section, we arrive at the following upper bound for the function $\phi(z)$ on the compact subset Y_ε of X.

Theorem 6.1.11. *With the recollections of Remark 6.1.3, for any $\alpha \in (0, \lambda_1)$ and $\delta \in (0, \widetilde{r}_\varepsilon)$, we have the upper bound*

$$\sup_{z\in Y_\varepsilon} |\phi(z)| \leq C_{\varepsilon,\alpha,\delta},$$

where the constant $C_{\varepsilon,\alpha,\delta}$ is given by

$$\begin{aligned} C_{\varepsilon,\alpha,\delta} =& \frac{B_{\varepsilon/2,\alpha,\delta}}{2g}\bigg(1+|\mathcal{P}| - \frac{|\mathcal{P}|\,C''_{\mathrm{par}}}{\log(\varepsilon/2)}\bigg) - \frac{2\,|\mathcal{P}|\log(\varepsilon/2)}{g} + \frac{C'_{\mathrm{par}}}{2g} + \\ & \frac{2\pi\big(d_X+1\big)^2}{\lambda_1\,\mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{2\pi|c_X-1|}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{C''_{\mathrm{par}}}{g\log(\varepsilon/2)}. \end{aligned} \tag{6.32}$$

Proof. From Corollary 6.1.2, we have

$$\begin{aligned} \phi(z) =& \frac{1}{2g}\bigg(H(z)+P(z) - \int_{Y_{\varepsilon/2}} P(\zeta)\,\mu_{\mathrm{shyp}}(\zeta)\bigg) + \\ & \frac{1}{2g}\sum_{p\in\mathcal{P}}\bigg(\int_{\partial U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)d^c_\zeta P(\zeta) - \int_{\partial U_{\varepsilon/2}(p)} P(\zeta)d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\bigg) + \\ & \frac{1}{8\pi g}\sum_{p\in\mathcal{P}}\int_{U_{\varepsilon/2}(p)} g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{8g^2} - \frac{2\pi(c_X-1)}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)}. \end{aligned} \tag{6.33}$$

Combining Propositions 6.1.4 and 6.1.5, the first line on the right-hand side of equation (6.33) is bounded by

$$\frac{1}{2g}\bigg(B_{\varepsilon/2,\alpha,\delta} - 2\,|\mathcal{P}|\log(\varepsilon/2)\bigg). \tag{6.34}$$

Combining Propositions 6.1.6 and 6.1.7, the second line on the right-hand side of equation (6.33) is bounded by

$$\frac{1}{2g}\bigg(|\mathcal{P}|\,B_{\varepsilon/2,\alpha,\delta} - 2|\mathcal{P}|\log(\varepsilon/2) + C'_{\mathrm{par}}\bigg). \tag{6.35}$$

Combining Propositions 6.1.8 and 6.1.9, the third line on the right-hand side of equation (6.33) is bounded by

$$-\frac{|\mathcal{P}|\,C''_{\mathrm{par}}}{4g\log(\varepsilon/2)}\bigg(B_{\varepsilon/2,\alpha,\delta}+\frac{4\pi}{\mathrm{vol_{hyp}}(X)}\bigg)+\frac{2\pi\,(d_X+1)^2}{\lambda_1\,\mathrm{vol_{hyp}}(X)}+\frac{2\pi|c_X-1|}{g\,\mathrm{vol_{hyp}}(X)}.$$

Using the inequality $\pi|\mathcal{P}|\leq \mathrm{vol_{hyp}}(X)$, we derive the upper bound for the first term in the above expression

$$-\frac{|\mathcal{P}|\,C''_{\mathrm{par}}}{4g\log(\varepsilon/2)}\bigg(B_{\varepsilon/2,\alpha,\delta}+\frac{4\pi}{\mathrm{vol_{hyp}}(X)}\bigg)\leq -\frac{|\mathcal{P}|\,C''_{\mathrm{par}}\,B_{\varepsilon/2,\alpha,\delta}}{2g\log(\varepsilon/2)}-\frac{C''_{\mathrm{par}}}{g\log(\varepsilon/2)}.$$

Using the above estimate, we have the upper bound for the third line on the right-hand side of equation (6.33)

$$-\frac{|\mathcal{P}|\,C''_{\mathrm{par}}\,B_{\varepsilon/2,\alpha,\delta}}{2g\log(\varepsilon/2)}-\frac{C''_{\mathrm{par}}}{g\log(\varepsilon/2)}+\frac{2\pi\,(d_X+1)^2}{\lambda_1\,\mathrm{vol_{hyp}}(X)}+\frac{2\pi|c_X-1|}{g\,\mathrm{vol_{hyp}}(X)}. \tag{6.36}$$

The proof of the theorem follows from combining the upper bounds derived in (6.34), (6.35), and (6.36), and rearranging the terms. □

In the following theorem, we obtain an upper bound for the difference of the canonical and hyperbolic Green's function on the compact subset Y_ε on X. It can be seen as an extension of Theorem 4.8 in [11] to the case when X is non-compact.

Theorem 6.1.12. *For any $\alpha\in(0,\lambda_1)$ and $\delta\in(0,\widetilde{r}_\varepsilon)$, we have the upper bound*

$$\sup_{z,w\in Y_\varepsilon}\big|g_{\mathrm{hyp}}(z,w)-g_{\mathrm{can}}(z,w)\big|\leq 2\,C_{\varepsilon,\alpha,\delta},$$

where $C_{\varepsilon,\alpha,\delta}$ is as in Theorem 6.1.11.

Proof. From Proposition 2.6.4, we have

$$\sup_{z,w\in Y_\varepsilon}\big|g_{\mathrm{hyp}}(z,w)-g_{\mathrm{can}}(z,w)\big|\leq \sup_{z\in Y_\varepsilon}2\,|\phi(z)|.$$

The proof of the theorem follows from the estimate of the function $\phi(z)$ given in Theorem 6.1.11. □

The following theorem is an extension of Theorem 4.9 in [11] to non-compact Riemann surfaces.

Theorem 6.1.13. *For any $\alpha\in(0,\lambda_1)$ and $\delta\in(0,\widetilde{r}_\varepsilon)$, we have the upper bound*

$$\sup_{z,w\in Y_\varepsilon}\bigg|g_{\mathrm{can}}(z,w)-\sum_{\gamma\in S_\Gamma(\delta;z,w)}g_{\mathbb{H}}(z,\gamma w)\bigg|\leq A_{\varepsilon,\alpha,\delta}, \tag{6.37}$$

where $A_{\varepsilon,\alpha,\delta}=B_{\varepsilon,\alpha,\delta}+2\,C_{\varepsilon,\alpha,\delta}$ with $B_{\varepsilon,\alpha,\delta}$ and $C_{\varepsilon,\alpha,\delta}$ as defined in Theorems 5.2.11 and 6.1.11, respectively.

Proof. From Theorem 5.2.11, for any $\alpha \in (0, \lambda_1)$ and $\delta \in (0, \widetilde{r}_\varepsilon)$, we have

$$\sup_{z,w \in Y_\varepsilon} \Big| g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) \Big| \leq B_{\varepsilon,\alpha,\delta}. \tag{6.38}$$

The claim follows from the elementary inequality

$$\Big| g_{\mathrm{can}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) \Big| \leq$$
$$\big| g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w) \big| + \Big| g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) \Big|,$$

and by combining the estimates obtained in Theorem 6.1.12 and (6.38). □

6.2 Bounds for the canonical Green's function at parabolic fixed points

In this section using results proven in [4], and Sections 5.3 and 6.1, we obtain estimates of the canonical Green's function in the neighborhoods of parabolic fixed points.

We first use results from [4] to estimate the function $\phi(z)$ at parabolic fixed points. We then use results from Sections 5.3 and 6.1, first to bound the difference of the hyperbolic and canonical Green's functions, and then the canonical Green's function, in the neighborhoods of parabolic fixed points.

In the following proposition, we describe the behavior of the function $\phi(z)$ at a parabolic fixed point.

Proposition 6.2.1. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, for any $z \in U_\varepsilon(p)$, we have the relation*

$$\phi(z) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log \big|\vartheta_p(z)\big|}{\log \varepsilon} \right) - \int_{U_\varepsilon(p)} g_{\varepsilon,p}(z,\zeta)\, \mu_{\mathrm{can}}(\zeta) + \phi_{\mathrm{har},p}(z),$$

where $g_{\varepsilon,p}(z,\zeta) = \log \left| \dfrac{(1/\varepsilon)\big(\vartheta_p(z) - \vartheta_p(\zeta)\big)}{1 - (1/\varepsilon^2)\vartheta_p(z)\overline{\vartheta_p(\zeta)}} \right|^2$ *and $\phi_{\mathrm{har},p}(z)$ is a harmonic function on $U_\varepsilon(p)$.*

Proof. The proposition follows from p. 86 of [4], after making the appropriate sign changes and adjusting the normalization used. □

Remark 6.2.2. Let $p \in \mathcal{P}$ be a parabolic fixed point. Then for $z, \zeta \in U_\varepsilon(p)$ and $z \neq \zeta$, the function $g_{\varepsilon,p}(z,\zeta)$ is a non-positive function and vanishes on the boundary $\partial U_\varepsilon(p)$, which implies that

$$\Big| \int_{U_\varepsilon(p)} g_{\varepsilon,p}(z,\zeta)\, \mu_{\mathrm{can}}(\zeta) \Big| = - \int_{U_\varepsilon(p)} g_{\varepsilon,p}(z,\zeta)\, \mu_{\mathrm{can}}(\zeta)$$

and for $z \in \partial U_{\varepsilon}(p)$, we have

$$\phi_{\mathrm{har},p}(z) = \phi(z).$$

Proposition 6.2.3. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, for any $\alpha \in (0, \lambda_1)$, $\delta \in (0, \widetilde{r}_{\varepsilon})$, and $z \in U_{\varepsilon}(p)$, we have the upper bound*

$$|\phi(z)| \leq \frac{4\pi}{\mathrm{vol_{hyp}}(X)} \log\left(\frac{\log\left|\vartheta_p(z)\right|}{\log\varepsilon}\right) + \frac{\pi d_X}{(\log\varepsilon)^2\,\mathrm{vol_{hyp}}(X)} + C_{\varepsilon,\alpha,\delta},$$

where d_X and $C_{\varepsilon,\alpha,\delta}$ are as defined in (1.5) and Theorem 6.1.11, respectively.

Proof. From Proposition 6.2.1, we have the elementary estimate

$$|\phi(z)| \leq \frac{4\pi}{\mathrm{vol_{hyp}}(X)} \log\left(\frac{\log\left|\vartheta_p(z)\right|}{\log\varepsilon}\right) + \left|\int_{U_{\varepsilon}(p)} g_{\varepsilon,p}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta)\right| + \max_{z\in U_{\varepsilon}(p)} \left|\phi_{\mathrm{har},p}(z)\right|. \tag{6.39}$$

As $\phi_{\mathrm{har},p}(z)$ is a harmonic function, $\left|\phi_{\mathrm{har},p}(z)\right|$ is a subharmonic function. So from the maximum principle of subharmonic functions, and from Theorem 6.1.11, it follows that

$$\max_{z\in U_{\varepsilon}(p)} |\phi_{\mathrm{har},p}(z)| = \max_{z\in \partial U_{\varepsilon}(p)} |\phi(z)| \leq \max_{z\in \partial Y_{\varepsilon}} |\phi(z)| \leq C_{\varepsilon,\alpha,\delta}. \tag{6.40}$$

Furthermore, from (3.6) on p. 87 of [4], we have the estimate

$$\left|\int_{U_{\varepsilon}(p)} g_{\varepsilon,p}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta)\right| = -\int_{U_{\varepsilon}(p)} g_{\varepsilon,p}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) \leq \frac{\pi d_X}{(\log\varepsilon)^2\,\mathrm{vol_{hyp}}(X)}. \tag{6.41}$$

The claimed estimate follows from combining (6.39), (6.40), and (6.41). □

In the following proposition, using Proposition 6.2.3, we derive an upper bound for the difference of the hyperbolic and canonical Green's functions, when one variable is in the neighborhood of a parabolic fixed point and the other is bounded away from the parabolic fixed points.

Proposition 6.2.4. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, for any $\alpha \in (0, \lambda_1)$, $\delta \in (0, \widetilde{r}_{\varepsilon})$, $z \in U_{\varepsilon}(p)$, and $w \in Y_{\varepsilon}$, we have the upper bound*

$$\left|g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)\right| \leq \frac{4\pi}{\mathrm{vol_{hyp}}(X)} \log\left(\frac{\log\left|\vartheta_p(z)\right|}{\log\varepsilon}\right) + \frac{\pi d_X}{(\log\varepsilon)^2\,\mathrm{vol_{hyp}}(X)} + 2C_{\varepsilon,\alpha,\delta},$$

where d_X and $C_{\varepsilon,\alpha,\delta}$ are as defined in (1.5) and Theorem 6.1.11, respectively.

Proof. For any $\alpha \in (0, \lambda_1)$, $\delta \in (0, \widetilde{r}_\varepsilon)$, $z \in U_\varepsilon(p)$, and $w \in Y_\varepsilon$, we have

$$\big|g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)\big| \leq |\phi(z)| + \sup_{w \in Y_\varepsilon} |\phi(w)|.$$

The claimed estimate follows from combining the estimates obtained in Theorem 6.1.11, and Proposition 6.2.3. □

In the following proposition, using Proposition 6.2.3, we derive an upper bound for the difference of the hyperbolic and canonical Green's functions, when both variables are in the neighborhoods of different parabolic fixed points.

Proposition 6.2.5. *Let $p, q \in \mathcal{P}$ be two parabolic fixed points with $p \neq q$. Then, for any $\alpha \in (0, \lambda_1)$, $\delta \in (0, \widetilde{r}_\varepsilon)$, $z \in U_\varepsilon(p)$, and $w \in U_\varepsilon(q)$, we have the upper bound*

$$\big|g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)\big| \leq \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log\big|\vartheta_p(z)\big|}{\log\varepsilon}\right) + \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log\big|\vartheta_p(w)\big|}{\log\varepsilon}\right) + \frac{2\pi d_X}{(\log\varepsilon)^2 \,\mathrm{vol}_{\mathrm{hyp}}(X)} + 2C_{\varepsilon,\alpha,\delta},$$

where d_X and $C_{\varepsilon,\alpha,\delta}$ are as defined in (1.5) and Theorem 6.1.11, respectively.

Proof. The claimed upper bound follows from the estimate obtained in Proposition 6.2.3. □

In the following proposition, using Proposition 6.2.3, we derive an upper bound for the difference of the hyperbolic and canonical Green's functions, when both variables are in the neighborhood of the same parabolic fixed point.

Proposition 6.2.6. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, for any $\alpha \in (0, \lambda_1)$, $\delta \in (0, \widetilde{r}_\varepsilon)$, and $z, w \in U_\varepsilon(p)$, we have the upper bound*

$$\big|g_{\mathrm{hyp}}(z,w) - g_{\mathrm{can}}(z,w)\big| \leq \frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log\big|\vartheta_p(z)\big|}{\log\varepsilon}\right) + \frac{2\pi d_X}{(\log\varepsilon)^2 \,\mathrm{vol}_{\mathrm{hyp}}(X)} + 2C_{\varepsilon,\alpha,\delta},$$

where d_X and $C_{\varepsilon,\alpha,\delta}$ are as defined in (1.5) and Theorem 6.1.11, respectively.

Proof. The claimed upper bound follows from the estimate obtained in Proposition 6.2.3. □

The following theorem is an extension of Theorem 6.1.13 to parabolic fixed points, when one variable is in the neighborhood of a parabolic fixed point and the other remains bounded away from the parabolic fixed points.

Theorem 6.2.7. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, for any $\alpha \in (0, \lambda_1)$ and $\delta \in (0, \widetilde{r}_\varepsilon)$, we have the upper bound*

$$\sup_{\substack{z \in U_\varepsilon(p) \\ w \in Y_\varepsilon}} \Big| g_{\mathrm{can}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) \Big| \leq A_{\varepsilon,\alpha,\delta} + \frac{\pi d_X}{(\log \varepsilon)^2 \operatorname{vol}_{\mathrm{hyp}}(X)},$$

where d_X and $A_{\varepsilon,\alpha,\delta}$ are as defined in (1.5) and Theorem 6.1.13, respectively.

Proof. From Proposition 2.6.4, we have

$$\begin{aligned} g_{\mathrm{can}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) = \\ g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) - \phi(z) - \phi(w). \end{aligned}$$

Combining Proposition 6.2.1 and Lemma 5.3.3 for $z \in U_\varepsilon(p)$ and $w \in Y_\varepsilon$, the right-hand side of the above equation can be expressed as

$$h_{\delta,p}(z,w) - \phi_{\mathrm{har},p}(z) + \int_{U_\varepsilon(p)} g_{\varepsilon,p}(z,\zeta)\, \mu_{\mathrm{can}}(\zeta) - \phi(w),$$

where $h_{\delta,p}(z,w)$ is as defined in Lemma 5.3.3. This implies that

$$\begin{aligned} \sup_{\substack{z \in U_\varepsilon(p) \\ w \in Y_\varepsilon}} \Big| g_{\mathrm{can}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) \Big| \leq \sup_{\substack{z \in U_\varepsilon(p) \\ w \in Y_\varepsilon}} \big| h_{\delta,p}(z,w) \big| + \\ \sup_{z \in U_\varepsilon(p)} \big| \phi_{\mathrm{har},p}(z) \big| + \sup_{z \in U_\varepsilon(p)} \Big| \int_{U_\varepsilon(p)} g_{\varepsilon,p}(z,\zeta)\, \mu_{\mathrm{can}}(\zeta) \Big| + \sup_{w \in Y_\varepsilon} \big| \phi(w) \big|. \end{aligned}$$

Combining the estimates obtained in Corollary 5.3.4 and (6.40), we arrive at

$$\sup_{\substack{z \in U_\varepsilon(p) \\ w \in Y_\varepsilon}} \big| h_{\delta,p}(z,w) \big| + \sup_{z \in U_\varepsilon(p)} \big| \phi_{\mathrm{har},p}(z) \big| \leq B_{\varepsilon,\alpha,\delta} + C_{\varepsilon,\alpha,\delta}, \tag{6.42}$$

for any $\alpha \in (0, \lambda_1)$ and $\delta \in (0, \widetilde{r}_\varepsilon)$. Combining the estimates obtained in (6.41) and Theorem 6.1.11, we find

$$\sup_{z \in U_\varepsilon(p)} \Big| \int_{U_\varepsilon(p)} g_{\varepsilon,p}(z,\zeta)\, \mu_{\mathrm{can}}(\zeta) \Big| + \sup_{w \in Y_\varepsilon} |\phi(w)| \leq \frac{\pi d_X}{(\log \varepsilon)^2 \operatorname{vol}_{\mathrm{hyp}}(X)} + C_{\varepsilon,\alpha,\delta}, \tag{6.43}$$

for any $\alpha \in (0, \lambda_1)$ and $\delta \in (0, \widetilde{r}_\varepsilon)$. The proof of the theorem follows from combining the upper bounds derived in (6.42) and (6.43). $\square$

The following theorem is an extension of Theorem 6.1.13 to parabolic fixed points, when both variables are in the neighborhoods of different parabolic fixed points.

Theorem 6.2.8. *Let $p, q \in \mathcal{P}$ be two parabolic fixed points with $p \neq q$. Then, for any $\alpha \in (0, \lambda_1)$ and $\delta \in (0, \widetilde{r}_\varepsilon)$, we have the upper bound*

$$\sup_{\substack{z \in U_\varepsilon(p) \\ w \in U_\varepsilon(q)}} \bigg| g_{\mathrm{can}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) \bigg| \leq A_{\varepsilon,\alpha,\delta} + \frac{2\pi d_X}{(\log\varepsilon)^2 \operatorname{vol}_{\mathrm{hyp}}(X)},$$

where d_X and $A_{\varepsilon,\alpha,\delta}$ are as defined in (1.5) and Theorem 6.1.13, respectively.

Proof. From Proposition 2.6.4, we have

$$\begin{aligned} &g_{\mathrm{can}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) = \\ &g_{\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) - \phi(z) - \phi(w). \end{aligned}$$

Combining Proposition 6.2.1 and Corollary 5.3.5 for $z \in U_\varepsilon(p)$ and $w \in U_\varepsilon(q)$, the right-hand side of the above equation can be expressed as

$$\begin{aligned} &h_{\delta,p,q}(z,w) - \phi_{\mathrm{har},p}(z) - \phi_{\mathrm{har},q}(w) + \\ &\int_{U_\varepsilon(p)} g_{\varepsilon,p}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) + \int_{U_\varepsilon(q)} g_{\varepsilon,q}(w,\zeta)\,\mu_{\mathrm{can}}(\zeta), \end{aligned}$$

where $h_{\delta,p,q}(z,w)$ is as defined in Corollary 5.3.5. This implies that

$$\begin{aligned} &\sup_{\substack{z \in U_\varepsilon(p) \\ w \in U_\varepsilon(q)}} \bigg| g_{\mathrm{can}}(z,w) - \sum_{\gamma \in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) \bigg| \leq \\ &\sup_{\substack{z \in U_\varepsilon(p) \\ w \in U_\varepsilon(q)}} \big| h_{\delta,p,q}(z,w) \big| + \sup_{z \in U_\varepsilon(p)} \big| \phi_{\mathrm{har},p}(z) \big| + \sup_{w \in U_\varepsilon(q)} \big| \phi_{\mathrm{har},q}(w) \big| + \\ &\sup_{z \in U_\varepsilon(p)} \bigg| \int_{U_\varepsilon(p)} g_{\varepsilon,p}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) \bigg| + \sup_{w \in U_\varepsilon(q)} \bigg| \int_{U_\varepsilon(q)} g_{\varepsilon,q}(w,\zeta)\,\mu_{\mathrm{can}}(\zeta) \bigg|. \end{aligned}$$

Combining the estimates obtained in Corollary 5.3.6 and (6.40), we arrive at

$$\begin{aligned} &\sup_{\substack{z \in U_\varepsilon(p) \\ w \in U_\varepsilon(q)}} \big| h_{\delta,p,q}(z,w) \big| + \sup_{z \in U_\varepsilon(p)} \big| \phi_{\mathrm{har},p}(z) \big| + \sup_{w \in U_\varepsilon(q)} \big| \phi_{\mathrm{har},q}(w) \big| \leq \\ &B_{\varepsilon,\alpha,\delta} + 2C_{\varepsilon,\alpha,\delta} = A_{\varepsilon,\alpha,\delta}. \end{aligned} \tag{6.44}$$

Furthermore, from the upper bound derived in (6.41), we obtain

$$\begin{aligned} &\sup_{z \in U_\varepsilon(p)} \bigg| \int_{U_\varepsilon(p)} g_{\varepsilon,p}(z,\zeta)\,\mu_{\mathrm{can}}(\zeta) \bigg| + \sup_{w \in U_\varepsilon(q)} \bigg| \int_{U_\varepsilon(q)} g_{\varepsilon,q}(w,\zeta)\,\mu_{\mathrm{can}}(\zeta) \bigg| \leq \\ &\frac{2\pi d_X}{(\log\varepsilon)^2 \operatorname{vol}_{\mathrm{hyp}}(X)}. \end{aligned} \tag{6.45}$$

The proof of the theorem follows from combining the estimates derived in (6.44) and (6.45). $\square$

Corollary 6.2.9. *Let $p, q \in P$ be two parabolic fixed points with $p \neq q$. Then, for any $\alpha \in (0, \lambda_1)$ and $\delta \in (0, \widetilde{r}_{\varepsilon})$, we have the upper bound*

$$\left| g_{\mathrm{can}}(p, q) \right| \leq A_{\varepsilon,\alpha,\delta} + \frac{2\pi d_X}{(\log \varepsilon)^2 \operatorname{vol}_{\mathrm{hyp}}(X)},$$

where d_X and $A_{\varepsilon,\alpha,\delta}$ are as defined in (1.5) and Theorem 6.1.13, respectively.

Proof. For $p, q \in \mathcal{P}$ with $p \neq q$, as $z, w \in X$ approach p, q, respectively, and for any $\gamma \in \Gamma$, the free-space Green's function $g_{\mathbb{H}}(z, \gamma w)$ approaches zero. Hence, the corollary follows immediately from Theorem 6.2.8 by letting $z \in U_{\varepsilon}(p)$ and $w \in U_{\varepsilon}(q)$ approach p and q, respectively. $\square$

Remark 6.2.10. The above bound for the canonical Green's function, when evaluated at two different parabolic fixed points is not optimal, as it depends on the choice of ε. We derive a sharper upper bound for it, independent of ε in next chapter.

Remark 6.2.11. Using our methods, we cannot extend Theorem 6.1.13 to the case when both the variables belong to the neighborhood of the same parabolic fixed point. This is because, for $p \in \mathcal{P}$ a parabolic fixed point and $z, w \in U_{\varepsilon}(p)$ with w approaching z, we find

$$g_{\mathrm{can}}(z, w) = -\log \left| \vartheta_z(w) \right|^2 + O_z(1) = -\log \left| \vartheta_p(z) - \vartheta_p(w) \right|^2 + O_z(1) =$$
$$4\pi \operatorname{Im}(\sigma_p^{-1} z) - \log |z - w|^2 + O_z(1),$$

where the contribution from the term $O_z(1)$ is a smooth function in z. This implies that the quantity

$$g_{\mathrm{can}}(z, w) - \sum_{\gamma \in S_{\Gamma}(\delta; z, w)} g_{\mathbb{H}}(z, \gamma w)$$

is not bounded, as w approaches z. Hence, an extension of Theorem 6.1.13 to the whole of X is not feasible.

Chapter 7

Applications

In the previous chapter, we have derived upper bounds for the canonical Green's function both away and in the neighborhoods of parabolic fixed points. Furthermore, in Corollary 6.2.9 we also computed an upper bound for the canonical Green's function when evaluated at two different parabolic fixed points. We also noted that the derived upper bound depends on the choice of an $\varepsilon > 0$.

In this chapter, using different techniques, we compute a sharper upper bound for the canonical Green's function when evaluated at two different parabolic fixed points. We then derive bounds for the canonical Green's function through covers and for families of modular curves.

7.1 Bounds for the canonical Green's function at parabolic fixed points, revisited

Let $p, q \in \mathcal{P}$ be two parabolic fixed points with $p \neq q$. Then, combining Proposition 2.6.4 and Theorem 4.3.8, we find

$$g_{\mathrm{can}}(p,q) = \lim_{z\to p}\lim_{w\to q}\big(g_{\mathrm{hyp}}(z,w) - \phi(z) - \phi(w)\big), \tag{7.1}$$

where

$$\phi(z) = \frac{H(z)}{2g} + \frac{1}{8\pi g}\int_X g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{8g^2} - \frac{2\pi(c_X-1)}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)}. \tag{7.2}$$

Recall that the functions $P(z)$ and $H(z)$ are as defined in Sections 4.2 and 4.3, respectively, and the constants C_{hyp} and c_X are as defined in (3.36) and (4.1), respectively.

From Proposition 2.4.1, we know the complete asymptotics of the hyperbolic Green's function $g_{\mathrm{hyp}}(z,w)$ at the parabolic fixed points. Though we know that the function $\phi(z)$ is $\log\log$-singular at parabolic fixed points from Corollary 2.6.5, we do not know its complete asymptotics.

From Proposition 4.3.3, we know the complete asymptotics of the function $H(z)$ at the parabolic fixed points. Furthermore, we have derived an estimate of the constant C_{hyp} in Proposition 6.1.9.

So if we can compute the asymptotics of the integral

$$\int_X g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)$$

at the parabolic fixed points, using (7.2), we can compute the asymptotics of the function $\phi(z)$ at the parabolic fixed points. Hence, using (7.1) we will be able to compute another upper bound for the canonical Green's function when evaluated at two different parabolic fixed points. This upper bound will be sharper than the one deduced in Corollary 6.2.9.

In the following two lemmas, we compute the zeroth Fourier coefficient of the automorphic Green's function and the hyperbolic Green's function, which will be useful for computing the asymptotics of the integral

$$\int_X g_{\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)$$

at the parabolic fixed points.

Notation 7.1.1. For the remaining part of the thesis, for $p\in\mathcal{P}$ a parabolic fixed point and $z\in\mathbb{H}$, we denote $\mathrm{Im}(\sigma_p^{-1}z)$ by y_p.

Lemma 7.1.2. *Let $p,q\in\mathcal{P}$ be two parabolic fixed points. Then, for z and $w=u+iv\in X$ with $y_p>v$ and $vy_p>1$, and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$, we have*

$$\int_0^1 g_{\mathrm{hyp},s}(z,\sigma_q w)du=\frac{4\pi v^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},q}(z,s)+\frac{4\pi\delta_{p,q}}{2s-1}\big(v^s y_p^{1-s}-v^{1-s}y_p^s\big),\quad (7.3)$$

where $\mathcal{E}_{\mathrm{par},q}(z,s)$ denotes the parabolic Eisenstein series associated to the parabolic fixed point $q\in\mathcal{P}$, which is as defined in Section 1.5. Furthermore, for $v>y_p$ and $vy_p>1$, and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$, we have

$$\int_0^1 g_{\mathrm{hyp},s}(z,\sigma_q w)du=\frac{4\pi v^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},q}(z,s).\quad (7.4)$$

Proof. For z and $w=u+iv\in X$ with $y_p>v$ and $vy_p>1$, and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$, combining Lemmas 5.1 and 5.2 of [8], we have

$$\int_0^1 g_{\mathrm{hyp},s}(z,\sigma_q w)du=$$
$$\frac{4\pi y_p^{1-s}}{2s-1}\big(\delta_{p,q}v^s+\alpha_{p,q}(s)v^{1-s}\big)+\frac{4\pi v^{1-s}}{2s-1}\sum_{n\neq 0}\alpha_{p,q}(n,s)W_s(n\sigma_p^{-1}z).$$

The expression on the right-hand side of the above equation can be rewritten as

$$\frac{4\pi v^{1-s}}{2s-1}\bigg(\delta_{p,q}y_p^s+\alpha_{p,q}(s)y_p^{1-s}+\sum_{n\neq 0}\alpha_{p,q}(n,s)W_s(n\sigma_p^{-1}z)\bigg)+$$
$$\frac{4\pi\delta_{p,q}}{2s-1}\big(v^s y_p^{1-s}-v^{1-s}y_p^s\big). \tag{7.5}$$

For $s\in\mathbb{C}$ and $\mathrm{Re}(s)>1$, recalling the Fourier expansion of the parabolic Eisenstein series $\mathcal{E}_{\mathrm{par},q}(z,s)$ described in Theorem 1.5.5, we get

$$\frac{4\pi v^{1-s}}{2s-1}\bigg(\delta_{p,q}y_p^s+\alpha_{p,q}(s)y_p^{1-s}+\sum_{n\neq 0}\alpha_{p,q}(n,s)W_s(n\sigma_p^{-1}z)\bigg)=$$
$$\frac{4\pi v^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},q}(z,s). \tag{7.6}$$

Combining equations (7.5) and (7.6) proves equation (7.3).

We now prove equation (7.4). For $v>y_p$ and $vy_p>1$, and $s\in\mathbb{C}$ with $\mathrm{Re}(s)>1$, combining Lemmas 5.1 and 5.2 of [8], we have

$$\int_0^1 g_{\mathrm{hyp},s}(z,\sigma_q w)du=$$
$$\frac{4\pi v^{1-s}}{2s-1}\bigg(\delta_{p,q}y_p^s+\alpha_{p,q}(s)y_p^{1-s}+\sum_{n\neq 0}\alpha_{p,q}(n,s)W_s(n\sigma_p^{-1}z)\bigg).$$

From equation (7.6), we derive that

$$\int_0^1 g_{\mathrm{hyp},s}(z,\sigma_q w)du=\frac{4\pi v^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},q}(z,s),$$

which proves equation (7.4), and completes the proof of the lemma. □

Lemma 7.1.3. *Let $p,q\in\mathcal{P}$ be two parabolic fixed points. Then, for z and $w=u+iv\in X$ with $y_p>v$ and $vy_p>1$, we have*

$$\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)du=4\pi\kappa_q(z)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}-\frac{4\pi\log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}+4\pi\delta_{p,q}(v-y_p), \tag{7.7}$$

where $\kappa_q(z)$ denotes Kronecker's limit function, which is as defined in Section 1.5. Furthermore, for $v>y_p$ and $vy_p>1$, we have

$$\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)du=4\pi\kappa_q(z)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}-\frac{4\pi\log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}. \tag{7.8}$$

Proof. We first prove equation (7.7), and then prove equation (7.8). Observe that

$$\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)du = \int_0^1 \lim_{s\to 1}\bigg(g_{\mathrm{hyp},s}(z,\sigma_q w) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)s}\cdot\frac{1}{s-1}\bigg)du =$$
$$\lim_{s\to 1}\bigg(\int_0^1 g_{\mathrm{hyp},s}(z,\sigma_q w)du - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\cdot\frac{1}{s-1}\bigg) + \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}. \tag{7.9}$$

For z and $w = u+iv \in X$ with $y_p > v$ and $vy_p > 1$, combining equations (7.3) and (7.9), we find

$$\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)du = \lim_{s\to 1}\bigg(\frac{4\pi v^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},q}(z,s) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\cdot\frac{1}{s-1}\bigg) +$$
$$\lim_{s\to 1}\frac{4\pi\delta_{p,q}}{2s-1}\big(v^s y_p^{1-s} - v^{1-s}y_p^s\big) + \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}. \tag{7.10}$$

From equation (2.8), we have

$$\lim_{s\to 1}\bigg(\frac{4\pi v^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},q}(z,s) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\cdot\frac{1}{s-1}\bigg) =$$
$$4\pi\kappa_q(z) - \frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi\log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}. \tag{7.11}$$

Substituting the above limit into equation (7.10) gives us

$$\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)du = 4\pi\kappa_q(z) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi\log v}{\mathrm{vol}_{\mathrm{hyp}}(X)} + 4\pi\delta_{p,q}(v-y_p),$$

which proves equation (7.7).

We now prove equation (7.8). For $v > y_p$ and $vy_p > 1$, combining equations (7.4) and (7.9), we find

$$\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)du =$$
$$\lim_{s\to 1}\bigg(\frac{4\pi v^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},q}(z,s) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\cdot\frac{1}{s-1}\bigg) + \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}. \tag{7.12}$$

Combining equations (7.12) and (7.11), we find

$$\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)du = 4\pi\kappa_q(z) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi\log v}{\mathrm{vol}_{\mathrm{hyp}}(X)},$$

which proves equation (7.8), and hence, completes the proof of the lemma. □

Remark 7.1.4. As the calculations that follow involve the functions $P_{\mathrm{gen},p}(w)$ and $\Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(w)$, we recall their definitions again. For any parabolic fixed point $p \in \mathcal{P}$, the functions $P_{\mathrm{gen},p}(w)$ and $\Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(w)$ are as defined in (4.3) and Lemma 4.2.8, respectively. Furthermore, for any $p, q \in \mathcal{P}$, the constant

$k_{q,p}(0)$ denotes the Fourier coefficient of Kronecker's limit function $\kappa_q(z)$ at the parabolic fixed point p, as stated in Theorem 1.5.3.

The constants C_{hyp} and c_X are as defined in (3.36) and (4.1), respectively. The constant d_X is as defined in (1.5) and λ_1 denotes the first non-zero eigenvalue of the hyperbolic Laplacian Δ_{hyp} on X.

Proposition 7.1.5. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, with the recollections of Remark 7.1.4, for z and $w = u + iv \in X$ with $y_p > 1$, we have the formal decomposition*

$$\int_X g_{\mathrm{hyp}}(z,w)\,\Delta_{\mathrm{hyp}}\,P(w)\,\mu_{\mathrm{hyp}}(w) =$$
$$\sum_{q\in\mathcal{P}}\int_0^{1/y_p}\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\sigma_q w)\frac{dudv}{v^2}+$$
$$\sum_{q\in\mathcal{P}}\int_{1/y_p}^{\infty}\left(4\pi\kappa_q(z)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}-\frac{4\pi\log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2}+$$
$$4\pi\int_{1/y_p}^{y_p}(v-y_p)\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},p}(\sigma_p w)\frac{dv}{v^2}. \tag{7.13}$$

Proof. From Lemma 4.2.8, we have

$$\Delta_{\mathrm{hyp}}\,P(w) = \sum_{q\in\mathcal{P}}\sum_{\eta\in\Gamma_q\backslash\Gamma}\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\eta w).$$

Furthermore, we also know that every term of the series $\Delta_{\mathrm{hyp}}\,P(w)$ is negative, and the series itself is absolutely and uniformly convergent. This implies that

$$\int_X g_{\mathrm{hyp}}(z,w)\,\Delta_{\mathrm{hyp}}\,P(w)\,\mu_{\mathrm{hyp}}(w) =$$
$$\sum_{q\in\mathcal{P}}\sum_{\eta\in\Gamma_q\backslash\Gamma}\int_{\mathcal{F}} g_{\mathrm{hyp}}(z,w)\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\eta w)\,\mu_{\mathrm{hyp}}(w), \tag{7.14}$$

where $\mathcal{F} \subseteq \mathbb{H}$ denotes a fixed fundamental domain of the Riemann surface X. After making the substitution $w \mapsto \eta^{-1}\sigma_q w$, from the Γ-invariance of $g_{\mathrm{hyp}}(z,w)$, and from the $\mathrm{PSL}_2(\mathbb{R})$-invariance of $\mu_{\mathrm{hyp}}(z)$, formally for $w = u + iv \in X$, we find

$$\sum_{q\in\mathcal{P}}\sum_{\eta\in\Gamma_q\backslash\Gamma}\int_{\mathcal{F}} g_{\mathrm{hyp}}(z,w)\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\eta w)\,\mu_{\mathrm{hyp}}(w) =$$
$$\sum_{q\in\mathcal{P}}\sum_{\eta\in\Gamma_q\backslash\Gamma}\int_{\sigma_q^{-1}\eta\mathcal{F}} g_{\mathrm{hyp}}(z,\sigma_q w)\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\sigma_q w)\,\mu_{\mathrm{hyp}}(w).$$

From Remark 5.1.3, formally the right-hand side of the above equation unfolds to give

$$\sum_{q\in\mathcal{P}}\int_0^{\infty}\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\sigma_q w)\frac{dudv}{v^2}. \tag{7.15}$$

From Lemma 4.2.8, for $q \in \mathcal{P}$ a parabolic fixed point, we have

$$\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) = 2\left(\frac{2\pi v}{\sinh(2\pi v)}\right)^2 - 2.$$

From the above equation, we infer that the function $\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)$ does not depend on u. So the expression in (7.15) further decomposes to give

$$\begin{aligned}
&\sum_{q\in\mathcal{P}} \int_0^{1/y_p}\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)\,\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)\frac{du dv}{v^2} + \\
&\sum_{q\in\mathcal{P}} \int_{1/y_p}^{y_p}\left(\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w) du\right)\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2} + \\
&\sum_{q\in\mathcal{P}} \int_{y_p}^{\infty}\left(\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w) du\right)\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2}.
\end{aligned} \tag{7.16}$$

Since in the second line of formula (7.16) we have $1/y_p < v < y_p$, we can apply equation (7.7), and rewrite the second line of formula (7.16) as

$$\begin{aligned}
&\sum_{q\in\mathcal{P}} \int_{1/y_p}^{y_p}\left(\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w) du\right)\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2} = \\
&\sum_{q\in\mathcal{P}} \int_{1/y_p}^{y_p}\left(4\pi\kappa_q(z) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi \log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2} + \\
&4\pi \int_{1/y_p}^{y_p} (v - y_p)\,\Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\sigma_p w)\frac{dv}{v^2}.
\end{aligned} \tag{7.17}$$

Since in the third line of formula (7.16) we have $v > y_p > 1/y_p$, we can apply equation (7.8), and rewrite the third line of formula (7.16) as

$$\begin{aligned}
&\sum_{q\in\mathcal{P}} \int_{y_p}^{\infty}\left(\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w) du\right)\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2} = \\
&\sum_{q\in\mathcal{P}} \int_{y_p}^{\infty}\left(4\pi\kappa_q(z) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi \log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2}.
\end{aligned} \tag{7.18}$$

The proof of the proposition follows from combining equations (7.17) and (7.18). □

Remark 7.1.6. The formal unfolding of the integral obtained in Proposition 7.1.5 translates into an equality of integrals, only if each of the three integrals on the right-hand side of equation (7.13) converges absolutely, which we prove in the lemmas that follow.

In the following lemma, we prove the absolute convergence of the first integral on the right-hand side of equation (7.13), and compute its asymptotics as $z \in X$ approaches a parabolic fixed point.

Lemma 7.1.7. *Let $p, q \in \mathcal{P}$ be two parabolic fixed points. Then, with the recollections of Remark 7.1.4, for $z \in X$ and $w = u + iv \in \mathbb{H}$, the integral*

$$\int_0^{1/y_p} \int_0^1 g_{\mathrm{hyp}}(z, \sigma_q w)\, \Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) \frac{dudv}{v^2}$$

converges absolutely. Furthermore as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, we have

$$\sum_{q \in \mathcal{P}} \int_0^{1/y_p} \int_0^1 g_{\mathrm{hyp}}(z, \sigma_q w)\, \Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) \frac{dudv}{v^2} = o_z(1), \tag{7.19}$$

where the contribution from the term $o_z(1)$ is a smooth function in z, which approaches zero, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$.

Proof. For $v \in \mathbb{R}_{>0}$, from the formula for the function $\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)$ from Lemma 4.2.8, we derive that

$$\frac{\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)}{v^2} = \frac{8\pi^2}{\sinh^2(2\pi v)} - \frac{2}{v^2}$$

remains bounded. So it suffices to show that the integral

$$\int_0^{1/y_p} \int_0^1 g_{\mathrm{hyp}}(z, \sigma_q w) dudv$$

converges absolutely. Let $\mathcal{I}$ denote the set $[0,1] \times [0, 1/y_p]$. We view the above integral as a real-integral on the compact subset $\mathcal{I} \subset \mathbb{R}^2$. The hyperbolic Green's function $g_{\mathrm{hyp}}(z, \sigma_q w)$ is at most log-singular on a measure zero subset of the interior points of $\mathcal{I}$. Furthermore, the hyperbolic Green's function $g_{\mathrm{hyp}}(z, \sigma_q w)$ is at most $\log\log$-singular on a measure zero subset of the boundary points of $\mathcal{I}$. Hence, it is absolutely integrable on $\mathcal{I}$. This implies that the integral

$$\int_0^{1/y_p} \int_0^1 g_{\mathrm{hyp}}(z, \sigma_q w)\, \Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) \frac{dudv}{v^2}$$

converges absolutely, and also proves the asymptotic relation asserted in equation (7.19). $\square$

In the following lemma, we prove the absolute convergence of the first two terms involved in the second integral on the right-hand side of equation (7.13), and compute their asymptotics as $z \in X$ approaches a parabolic fixed point.

Lemma 7.1.8. *Let $p, q \in \mathcal{P}$ be two parabolic fixed points. Then, with the recollections of Remark 7.1.4, for $z \in X$ and $w = u + iv \in \mathbb{H}$, the integral*

$$\int_{1/y_p}^{\infty} \left(4\pi \kappa_q(z) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \right) \Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) \frac{dv}{v^2} \tag{7.20}$$

converges absolutely. Furthermore, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, we have

$$\sum_{q\in\mathcal{P}}\int_{1/y_p}^{\infty}\bigg(4\pi\kappa_q(z)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)\Delta_{\mathrm{hyp}}P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2}=$$
$$16\pi^2\bigg(-y_p+\frac{|\mathcal{P}|\big(\log y_p+1\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)}-\sum_{q\in\mathcal{P}}k_{q,p}(0)+\frac{2\pi}{3}\bigg)+O\bigg(\frac{\log y_p}{y_p}\bigg).$$

Proof. Substituting the formula for the function $\Delta_{\mathrm{hyp}}P_{\mathrm{gen},q}(\sigma_q w)$ from Lemma 4.2.8, we have

$$\int_{1/y_p}^{\infty}\bigg(4\pi\kappa_q(z)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)\Delta_{\mathrm{hyp}}P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2}=$$
$$\bigg(8\pi\kappa_q(z)-\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)\int_{1/y_p}^{\infty}\bigg(\bigg(\frac{2\pi v}{\sinh(2\pi v)}\bigg)^2-1\bigg)\frac{dv}{v^2}.$$

The integral on the right-hand side of the above equation further simplifies to give

$$\bigg(8\pi\kappa_q(z)-\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)\bigg[\frac{1}{v}-2\pi\coth(2\pi v)\bigg]_{1/y_p}^{\infty}=$$
$$\bigg(8\pi\kappa_q(z)-\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)\bigg(-2\pi-y_p+2\pi\coth\bigg(\frac{2\pi}{y_p}\bigg)\bigg). \tag{7.21}$$

Hence, from equation (7.21), we can conclude that the integral (7.20) converges absolutely.

We now compute the asymptotics of the expression obtained on the right-hand side of equation (7.21), as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$.

We first compute the asymptotics for the expression in the second bracket on the right-hand side of equation (7.21), as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$.

For $t \in \mathbb{R}_{>0}$, recall that the Taylor series expansion of the function $\coth(t)$ as t approaches zero is of the form

$$\coth(t)=\frac{1}{t}+\frac{t}{3}+O(t^3).$$

As $z \in X$ approaches $p \in \mathcal{P}$, the quantity $1/y_p$ approaches zero. So as $z \in X$ approaches $p \in \mathcal{P}$, using the Taylor expansion of $\coth(2\pi/y_p)$, we have the asymptotic relation

$$-2\pi-y_p+2\pi\coth\bigg(\frac{2\pi}{y_p}\bigg)=$$
$$-2\pi-y_p+2\pi\bigg(\frac{y_p}{2\pi}+\frac{2\pi}{3y_p}+O\bigg(\frac{1}{y_p^3}\bigg)\bigg)=-2\pi+\frac{4\pi^2}{3y_p}+O\bigg(\frac{1}{y_p^3}\bigg). \tag{7.22}$$

As $z \in X$ approaches $p \in \mathcal{P}$, from the Fourier expansion of Kronecker's limit function $\kappa_q(z)$ described in Theorem 1.5.3, we have the asymptotic relation for the first bracket on the right-hand side of equation (7.21)

$$\begin{aligned}&8\pi\kappa_q(z) - \frac{8\pi}{\mathrm{vol_{hyp}}(X)} = \\ &8\pi\delta_{p,q}y_p - \frac{8\pi\log y_p}{\mathrm{vol_{hyp}}(X)} + 8\pi k_{q,p}(0) - \frac{8\pi}{\mathrm{vol_{hyp}}(X)} + O\big(e^{-2\pi y_p}\big).\end{aligned} \tag{7.23}$$

Combining equations (7.22) and (7.23), as $z \in X$ approaches $p \in \mathcal{P}$, we have the asymptotic relation for the right-hand side of equation (7.21)

$$\begin{aligned}&\bigg(8\pi\delta_{p,q}y_p - \frac{8\pi\log y_p}{\mathrm{vol_{hyp}}(X)} + 8\pi k_{q,p}(0) - \frac{8\pi}{\mathrm{vol_{hyp}}(X)} + O\big(e^{-2\pi y_p}\big)\bigg)\times \\ &\qquad\qquad\qquad\bigg(-2\pi + \frac{4\pi^2}{3y_p} + O\Big(\frac{1}{y_p^3}\Big)\bigg) = \\ &16\pi^2\bigg(-\delta_{p,q}y_p + \frac{(\log y_p + 1)}{\mathrm{vol_{hyp}}(X)} - k_{q,p}(0) + \frac{2\pi}{3}\delta_{p,q} + O\Big(\frac{\log y_p}{y_p}\Big)\bigg).\end{aligned}$$

Hence, as $z \in X$ approaches $p \in \mathcal{P}$, we derive

$$\begin{aligned}&\sum_{q\in\mathcal{P}}\int_{1/y_p}^{\infty}\bigg(4\pi\kappa_q(z) - \frac{4\pi}{\mathrm{vol_{hyp}}(X)}\bigg)\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2} = \\ &16\pi^2\bigg(-y_p + \frac{|\mathcal{P}|\big(\log y_p + 1\big)}{\mathrm{vol_{hyp}}(X)} - \sum_{q\in\mathcal{P}}k_{q,p}(0) + \frac{2\pi}{3}\bigg) + O\Big(\frac{\log y_p}{y_p}\Big),\end{aligned}$$

which completes the proof of the lemma. □

In the following lemma, we prove the absolute convergence of the third term involved in the second integral on the right-hand side of equation (7.13), and compute an upper bound for it.

Lemma 7.1.9. *Let $p, q \in \mathcal{P}$ be two parabolic fixed points. Then, with the recollections of Remark 7.1.4, for $z \in X$ and $w = u + iv \in \mathbb{H}$, the integral*

$$\int_{1/y_p}^{\infty}\frac{4\pi\log v}{\mathrm{vol_{hyp}}(X)}\,\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2} \tag{7.24}$$

converges absolutely. Furthermore, we have the upper bound

$$\sum_{q\in\mathcal{P}}\int_{1/y_p}^{\infty}\bigg|\frac{4\pi\log v}{\mathrm{vol_{hyp}}(X)}\,\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},q}(\sigma_q w)\bigg|\frac{dv}{v^2} \leq \frac{8\pi|\mathcal{P}|}{\mathrm{vol_{hyp}}(X)}\bigg(1 + \frac{4\pi^2}{3}\bigg). \tag{7.25}$$

Proof. We prove the upper bound asserted in (7.25), which also proves the absolute convergence of the integral in (7.24). Observing the elementary estimate

$$\begin{aligned}&\int_{1/y_p}^{\infty}\Big|\log v\,\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},q}(\sigma_q w)\Big|\frac{dv}{v^2} \leq \\ &\int_0^1\Big|\log v\,\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},q}(\sigma_q w)\Big|\frac{dv}{v^2} + \int_1^{\infty}\Big|\log v\,\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},q}(\sigma_q w)\Big|\frac{dv}{v^2},\end{aligned} \tag{7.26}$$

we proceed to bound the two integrals on the right-hand side of the above inequality. For $v \in \mathbb{R}_{>0}$, from the formula stated in Lemma 4.2.8 for the function $\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)$, we find that the function

$$-\frac{\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)}{v^2} = \frac{2}{v^2} - \frac{8\pi^2}{\sinh^2(2\pi v)}$$

is a positive monotone decreasing function, and hence, attains its maximum value at $v = 0$. So we compute the limit

$$-\lim_{v\to 0} \frac{\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)}{v^2}.$$

For $t \in \mathbb{R}_{>0}$, the Taylor expansion for the function $1/\sinh^2(t)$ as t approaches zero is of the form

$$\frac{1}{\sinh^2(t)} = \frac{1}{t^2} - \frac{1}{3} + O(t^2),$$

from which we compute

$$\max_{v\in\mathbb{R}_{>0}} \left| \frac{\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)}{v^2} \right| = -\lim_{v\to 0} \frac{\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)}{v^2} =$$
$$\lim_{v\to 0} \left(\frac{2}{v^2} - 8\pi^2 \left(\frac{1}{4\pi^2 v^2} - \frac{1}{3} \right) \right) = \frac{8\pi^2}{3}.$$

So using the fact that, for $v \in (0,1]$, $|\log v| = -\log v$, we have the upper bound for the first integral on the right-hand side of inequality (7.26)

$$\int_0^1 \left| \log v \, \Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) \right| \frac{dv}{v^2} \leq -\frac{8\pi^2}{3} \int_0^1 \log v dv = \frac{8\pi^2}{3}. \tag{7.27}$$

We now bound the second integral on the right-hand side of inequality (7.26). From the formula stated in Lemma 4.2.8 for the function $\Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w)$, we have the upper bound

$$\max_{v\in\mathbb{R}_{>0}} \left| \Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) \right| = \max_{v\in\mathbb{R}_{>0}} \left(2 - \frac{8\pi^2 v^2}{\sinh^2(2\pi v)} \right) = 2.$$

Using the above bound, we derive the estimate of the second integral on the right-hand side of inequality (7.26)

$$\int_1^\infty \left| \log v \, \Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) \right| \frac{dv}{v^2} \leq$$
$$2\int_1^\infty \frac{\log v}{v^2} dv = 2\left(\left[-\frac{\log v}{v} \right]_1^\infty + \left[-\frac{1}{v} \right]_1^\infty \right) = 2. \tag{7.28}$$

Hence combining the estimates derived in (7.27) and (7.28), we arrive at the upper bound

$$\sum_{q\in\mathcal{P}} \int_{1/y_p}^\infty \left| \frac{4\pi \log v}{\mathrm{vol}_{\mathrm{hyp}}(X)} \Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) \right| \frac{dv}{v^2} \leq \frac{8\pi |P|}{\mathrm{vol}_{\mathrm{hyp}}(X)} \left(1 + \frac{4\pi^2}{3} \right),$$

which proves the lemma. $\square$

In the following lemma, we prove the absolute convergence of the second term involved in the third integral on the right-hand side of equation (7.13), and compute its asymptotics as $z \in X$ approaches a parabolic fixed point.

Lemma 7.1.10. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, with the recollections of Remark 7.1.4, for $z \in X$ and $w = u + iv \in \mathbb{H}$ with $y_p > 1$, the integral*

$$-4\pi y_p \int_{1/y_p}^{y_p} \Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\sigma_p w) \frac{dv}{v^2} \tag{7.29}$$

converges absolutely. Furthermore, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, we have

$$\begin{aligned} -4\pi y_p \int_{1/y_p}^{y_p} \Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\sigma_p w) \frac{dv}{v^2} &= \\ 4\pi \bigg(4\pi y_p \coth(2\pi y_p) - 2 - \frac{8\pi^2}{3} \bigg) &+ O\bigg(\frac{1}{y_p^2}\bigg). \end{aligned} \tag{7.30}$$

Proof. Using the formula for the function $\Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\sigma_p w)$ from Lemma 4.2.8 for a parabolic fixed point $p \in \mathcal{P}$, we find

$$\begin{aligned} &-4\pi y_p \int_{1/y_p}^{y_p} \Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\sigma_p w) \frac{dv}{v^2} = \\ &-8\pi y_p \int_{1/y_p}^{y_p} \bigg(\frac{4\pi^2}{\sinh^2(2\pi v)} - \frac{1}{v^2} \bigg) dv = -8\pi y_p \bigg[\frac{1}{v} - 2\pi \coth(2\pi v) \bigg]_{1/y_p}^{y_p} = \\ &-8\pi y_p \bigg(\frac{1}{y_p} - 2\pi \coth(2\pi y_p) - y_p + 2\pi \coth\left(\frac{2\pi}{y_p}\right) \bigg). \end{aligned} \tag{7.31}$$

The right-hand side of equation (7.31) further simplifies to

$$-8\pi + 16\pi^2 y_p \coth(2\pi y_p) - 8\pi y_p \bigg(- y_p + 2\pi \coth\left(\frac{2\pi}{y_p}\right) \bigg). \tag{7.32}$$

This implies that the integral (7.29) converges absolutely.

We now investigate the asymptotic behavior of the third term in the above expression, at the parabolic fixed point $p \in \mathcal{P}$. As $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, from the Taylor expansion of $\coth(2\pi/y_p)$ already used in equation (7.22), we get

$$\begin{aligned} -8\pi y_p \bigg(- y_p + 2\pi \coth\left(\frac{2\pi}{y_p}\right) \bigg) &= -8\pi y_p \bigg(\frac{4\pi^2}{3y_p} + O\bigg(\frac{1}{y_p^3}\bigg) \bigg) \\ &= -\frac{32\pi^3}{3} + O\bigg(\frac{1}{y_p^2}\bigg). \end{aligned}$$

As $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, substituting the above asymptotic relation into (7.32), we get

$$-4\pi y_p \int_{1/y_p}^{y_p} \Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\sigma_p w) \frac{dv}{v^2} =$$
$$4\pi\bigg(4\pi y_p \coth(2\pi y_p) - 2 - \frac{8\pi^2}{3}\bigg) + O\bigg(\frac{1}{y_p^2}\bigg),$$

which proves the lemma. $\square$

In the following lemma, we prove the absolute convergence of the first term involved in the third integral on the right-hand side of equation (7.13), and compute its asymptotics as $z \in X$ approaches a parabolic fixed point.

Lemma 7.1.11. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, with the recollections of Remark 7.1.4, for $z \in X$ and $w = u + iv \in \mathbb{H}$ with $y_p > 1$, the integral*

$$4\pi \int_{1/y_p}^{y_p} \Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\sigma_p w) \frac{dv}{v}. \tag{7.33}$$

converges absolutely. Furthermore, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, we have

$$4\pi \int_{1/y_p}^{y_p} \Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\sigma_p w) \frac{dv}{v} = -8\pi \log y_p + 8\pi(1 - \log(4\pi)) + O\bigg(\frac{1}{y_p}\bigg). \tag{7.34}$$

Proof. Using the formula for the function $\Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\sigma_p w)$ from Lemma 4.2.8 for a parabolic fixed point $p \in \mathcal{P}$, we find

$$\int_{1/y_p}^{y_p} \Delta_{\mathrm{hyp}} P_{\mathrm{gen},p}(\sigma_p w) \frac{dv}{v} = 2 \int_{1/y_p}^{y_p} \bigg(-\frac{1}{v} + \frac{4\pi^2 v}{\sinh^2(2\pi v)} \bigg) dv =$$
$$2\bigg[-\log v - 2\pi v \coth(2\pi v) + \log(\sinh(2\pi v))\bigg]_{1/y_p}^{y_p}.$$

Substituting the formulae for $\coth(2\pi v)$ and $\sinh(2\pi v)$, the right-hand side of the above equation can be further simplified to

$$2\bigg[-\log v - 4\pi v - \frac{4\pi v}{e^{4\pi v} - 1} + \log\bigg(\frac{e^{4\pi v} - 1}{2}\bigg)\bigg]_{1/y_p}^{y_p}.$$

Observe that

$$\bigg[-\log v - 4\pi v - \frac{4\pi v}{e^{4\pi v} - 1} + \log\bigg(\frac{e^{4\pi v} - 1}{2}\bigg)\bigg]_{1/y_p}^{y_p} =$$
$$-\log y_p - 4\pi y_p - \frac{4\pi y_p}{e^{4\pi y_p} - 1} + \log\big(e^{4\pi y_p} - 1\big) +$$
$$\log\bigg(\frac{1}{y_p}\bigg) + \frac{4\pi}{y_p} + \frac{4\pi}{y_p\big(e^{4\pi/y_p} - 1\big)} - \log\big(e^{4\pi/y_p} - 1\big).$$

The right-hand side of the above equation further simplifies to

$$- \log y_p - \log \left(\frac{e^{4\pi y_p}}{e^{4\pi y_p} - 1} \right) - \frac{4\pi y_p}{e^{4\pi y_p} - 1} + \frac{4\pi}{y_p} + \\ \frac{4\pi}{y_p\big(e^{4\pi/y_p} - 1\big)} - \log \big(y_p\big(e^{4\pi/y_p} - 1\big)\big), \tag{7.35}$$

which proves that the integral (7.33) converges absolutely.

To prove (7.34), it suffices to compute the asymptotic expansion of each of the terms in the above expression, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$. As $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, we have the asymptotic relation for the first and second terms of (7.35)

$$- \log y_p - \log \left(\frac{e^{4\pi y_p}}{e^{4\pi y_p} - 1} \right) = - \log y_p + O\left(e^{-4\pi y_p}\right); \tag{7.36}$$

the third and fourth terms of (7.35) satisfy the asymptotic relation

$$- \frac{4\pi y_p}{e^{4\pi y_p} - 1} + \frac{4\pi}{y_p} = O\bigg(\frac{1}{y_p}\bigg). \tag{7.37}$$

As $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, the fifth term satisfies the asymptotic relation

$$\frac{4\pi}{y_p\big(e^{4\pi/y_p} - 1\big)} = \frac{4\pi}{y_p\bigg(\displaystyle\sum_{n=1}^{\infty} \frac{(4\pi)^n}{n!\, y_p^n}\bigg)} = \\ \frac{1}{1 + \displaystyle\sum_{n=1}^{\infty} \frac{(4\pi)^n}{(n+1)!\, y_p^n}} = 1 + O\bigg(\frac{1}{y_p}\bigg); \tag{7.38}$$

the sixth term satisfies the asymptotic relation

$$- \log \big(y_p\big(e^{4\pi/y_p} - 1\big)\big) = - \log \left(\sum_{n=1}^{\infty} \frac{(4\pi)^n}{n!\, y_p^{n-1}} \right) = \\ - \log \left(4\pi + \sum_{n=1}^{\infty} \frac{(4\pi)^{n+1}}{(n+1)!\, y_p^n} \right) = - \log(4\pi) + O\bigg(\frac{1}{y_p}\bigg). \tag{7.39}$$

Substituting the asymptotic relations obtained in equations (7.36), (7.37), (7.38), and (7.39) into (7.35), we derive the asymptotic relation

$$4\pi \int_{1/y_p}^{y_p} \Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},p}(\sigma_p w) \frac{dv}{v} = -8\pi \log y_p + 8\pi(1 - \log(4\pi)) + O\bigg(\frac{1}{y_p}\bigg),$$

as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$. This proves the lemma. $\square$

In the following proposition, combining all the asymptotics established in this section, we compute the asymptotics of the integral

$$\int_X g_{\mathrm{hyp}}(z,w)\,\Delta_{\mathrm{hyp}}\,P(w)\,\mu_{\mathrm{hyp}}(w),$$

as $z\in X$ approaches a parabolic fixed point $p\in\mathcal{P}$.

Proposition 7.1.12. *Let $p\in\mathcal{P}$ be a parabolic fixed point. Then, with the recollections of Remark 7.1.4, as $z\in X$ approaches the parabolic fixed point $p\in\mathcal{P}$, we have*

$$\int_X g_{\mathrm{hyp}}(z,w)\,\Delta_{\mathrm{hyp}}\,P(w)\,\mu_{\mathrm{hyp}}(w)=-\frac{32\pi^2(g-1)\log y_p}{\mathrm{vol}_{\mathrm{hyp}}(X)}-$$
$$\sum_{q\in\mathcal{P}}\int_{1/y_p}^{\infty}\frac{4\pi\log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2}+\alpha_p+o_z(1),$$

where

$$\alpha_p=\frac{16\pi^2|\mathcal{P}|}{\mathrm{vol}_{\mathrm{hyp}}(X)}-16\pi^2\sum_{q\in\mathcal{P}}k_{q,p}(0)-8\pi\log(4\pi),\tag{7.40}$$

and the contribution from the term $o_z(1)$ is a smooth function in z, which approaches zero, as $z\in X$ approaches the parabolic fixed point $p\in\mathcal{P}$.

Proof. As $z\in X$ approaches the parabolic fixed point $p\in\mathcal{P}$, from Proposition 7.1.5, we formally have

$$\int_X g_{\mathrm{hyp}}(z,w)\,\Delta_{\mathrm{hyp}}\,P(w)\,\mu_{\mathrm{hyp}}(w)=$$
$$\sum_{q\in\mathcal{P}}\int_0^{1/y_p}\int_0^1 g_{\mathrm{hyp}}(z,\sigma_q w)\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\sigma_q w)\frac{du dv}{v^2}+$$
$$\sum_{q\in\mathcal{P}}\int_{1/y_p}^{\infty}\bigg(4\pi\kappa_q(z)-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}-\frac{4\pi\log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2}+$$
$$4\pi\int_{1/y_p}^{y_p}(v-y_p)\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},p}(\sigma_p w)\frac{dv}{v^2}.\tag{7.41}$$

From Lemmas 7.1.7, 7.1.8, 7.1.9, 7.1.10, and 7.1.11, it follows that each of the integrals on the right-hand side of the above equation is absolutely convergent. This implies that the above equality of integrals indeed holds true for all $z\in X$ provided that $y_p>1$.

As $z\in X$ approaches the parabolic fixed point $p\in\mathcal{P}$, combining Lemmas 7.1.7 and 7.1.8, we find that the first two integrals on the right-hand side of equation (7.41) yield

$$16\pi^2\bigg(-y_p+\frac{|\mathcal{P}|\big(\log y_p+1\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)}-\sum_{q\in\mathcal{P}}k_{q,p}(0)+\frac{2\pi}{3}\bigg)-$$
$$\sum_{q\in\mathcal{P}}\int_{1/y_p}^{\infty}\frac{4\pi\log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},q}(\sigma_q w)\frac{dv}{v^2}+o_z(1),\tag{7.42}$$

where the contribution from the term $o_z(1)$ is a smooth function in z, which approaches zero, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$. As $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, combining Lemmas 7.1.10 and 7.1.11, we find that the third integral on the right-hand side of equation (7.41) yields

$$16\pi^2 y_p \coth(2\pi y_p) - 8\pi \log y_p - \frac{32\pi^3}{3} - 8\pi \log(4\pi) + O\bigg(\frac{1}{y_p}\bigg). \tag{7.43}$$

Combining (7.42) and (7.43), as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, the right-hand side of equation (7.41) simplifies to

$$\begin{aligned}&-16\pi^2 y_p + 16\pi^2 y_p \coth(2\pi y_p) + \frac{16\pi^2|\mathcal{P}|\log y_p}{\mathrm{vol}_{\mathrm{hyp}}(X)} - 8\pi \log y_p + \frac{16\pi^2|\mathcal{P}|}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \\ &16\pi^2 \sum_{q\in\mathcal{P}} k_{q,p}(0) - \sum_{q\in\mathcal{P}} \int_{1/y_p}^{\infty} \frac{4\pi \log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}\, \Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) \frac{dv}{v^2} - \\ &8\pi \log(4\pi) + o_z(1).\end{aligned} \tag{7.44}$$

As $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, we have the asymptotic relation for the first two terms in the above expression

$$\begin{aligned}&16\pi^2 y_p\big(\coth(2\pi y_p) - 1\big) = 16\pi^2 y_p\bigg(\frac{\cosh(2\pi y_p) - \sinh(2\pi y_p)}{\sinh(2\pi y_p)}\bigg) = \\ &16\pi^2 y_p\bigg(\frac{2e^{-2\pi y_p}}{e^{2\pi y_p} - e^{-2\pi y_p}}\bigg) = O\big(e^{-y_p}\big).\end{aligned}$$

Furthermore, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, the third and fourth terms in the above expression give

$$\frac{16\pi^2|\mathcal{P}|\log y_p}{\mathrm{vol}_{\mathrm{hyp}}(X)} - 8\pi \log y_p = -\frac{32\pi^2(g-1)\log y_p}{\mathrm{vol}_{\mathrm{hyp}}(X)}.$$

Hence, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, the expression in (7.44) further reduces to give

$$\begin{aligned}&-\frac{32\pi^2(g-1)\log y_p}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \sum_{q\in\mathcal{P}} \int_{1/y_p}^{\infty} \frac{4\pi \log v}{\mathrm{vol}_{\mathrm{hyp}}(X)}\, \Delta_{\mathrm{hyp}} P_{\mathrm{gen},q}(\sigma_q w) \frac{dv}{v^2} + \\ &\frac{16\pi^2|\mathcal{P}|}{\mathrm{vol}_{\mathrm{hyp}}(X)} - 16\pi^2 \sum_{q\in\mathcal{P}} k_{q,p}(0) - 8\pi\log(4\pi) + o_z(1).\end{aligned}$$

This completes the proof of the proposition. □

In the following corollary, using the above proposition, we compute the asymptotics of the function $\phi(z)$, as $z \in X$ approaches a parabolic fixed point $p \in \mathcal{P}$.

Corollary 7.1.13. *Let $p \in \mathcal{P}$ be a parabolic fixed point. Then, with the recollections of Remark 7.1.4, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, we have*

$$\phi(z) = -\frac{4\pi \log y_p}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \sum_{q \in \mathcal{P}} \int_{1/y_p}^{\infty} \frac{\log v}{2g\, \mathrm{vol}_{\mathrm{hyp}}(X)}\, \Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},q}(\sigma_q w) \frac{dv}{v^2} +$$
$$\frac{\alpha_p}{8\pi g} + \frac{2\pi k_{p,p}(0)}{g} - \frac{C_{\mathrm{hyp}}}{8g^2} - \frac{2\pi c_X}{g\, \mathrm{vol}_{\mathrm{hyp}}(X)} + o_z(1),$$

where the constant α_p is as defined in (7.40), and the contribution from the term $o_z(1)$ is a smooth function in z, which approaches zero, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$.

Proof. From Theorem 4.3.8, we have

$$\phi(z) = \frac{H(z)}{2g} + \frac{1}{8\pi g} \int_X g_{\mathrm{hyp}}(z,\zeta)\, \Delta_{\mathrm{hyp}}\, P(\zeta)\, \mu_{\mathrm{hyp}}(\zeta) - \frac{C_{\mathrm{hyp}}}{8g^2} - \frac{2\pi(c_X - 1)}{g\, \mathrm{vol}_{\mathrm{hyp}}(X)}.$$

As $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, we have from Proposition 4.3.3

$$\frac{H(z)}{2g} - \frac{C_{\mathrm{hyp}}}{8g^2} - \frac{2\pi(c_X - 1)}{g\, \mathrm{vol}_{\mathrm{hyp}}(X)} =$$
$$-\frac{4\pi \log y_p}{g\, \mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{2\pi k_{p,p}(0)}{g} - \frac{C_{\mathrm{hyp}}}{8g^2} - \frac{2\pi c_X}{g\, \mathrm{vol}_{\mathrm{hyp}}(X)} + O\Big(\frac{1}{y_p}\Big). \tag{7.45}$$

As $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$, using Proposition 7.1.12, we find that

$$\frac{1}{8\pi g} \int_X g_{\mathrm{hyp}}(z,\zeta)\, \Delta_{\mathrm{hyp}}\, P(\zeta)\, \mu_{\mathrm{hyp}}(\zeta) = -\frac{4\pi \log y_p}{\mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{4\pi \log y_p}{g\, \mathrm{vol}_{\mathrm{hyp}}(X)} -$$
$$\sum_{q \in \mathcal{P}} \int_{1/y_p}^{\infty} \frac{\log v}{2g\, \mathrm{vol}_{\mathrm{hyp}}(X)}\, \Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},q}(\sigma_q w) \frac{dv}{v^2} + \frac{\alpha_p}{8\pi g} + o_z(1), \tag{7.46}$$

where the contribution from the term $o_z(1)$ is a smooth function in z, which approaches zero, as $z \in X$ approaches the parabolic fixed point $p \in \mathcal{P}$. The proof of the corollary follows from combining the estimates deduced in equations (7.45) and (7.46). □

In the following theorem, using the above corollary, we compute an upper bound for the canonical Green's function, when evaluated at two different parabolic fixed points.

Theorem 7.1.14. *Let $p, q \in \mathcal{P}$ be two parabolic fixed points with $p \neq q$. Then, with the recollections of Remark 7.1.4, we have the upper bound*

$$\big|g_{\mathrm{can}}(p,q)\big| \leq 4\pi \big|k_{p,q}(0)\big| + \frac{2\pi}{g}\bigg(\sum_{\substack{s \in \mathcal{P}\\ s \neq p}} \big|k_{s,p}(0)\big| + \sum_{\substack{s \in \mathcal{P}\\ s \neq q}} \big|k_{s,q}(0)\big|\bigg) +$$
$$\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg(\frac{4\pi(d_X+1)^2}{\lambda_1} + \frac{\big|4\pi c_X\big|}{g} + \frac{43|\mathcal{P}|}{g} + 4\pi\bigg) + \frac{2\log(4\pi)}{g}.$$

Proof. For $z, w \in X$, we have from Proposition 2.6.4

$$g_{\mathrm{can}}(p,q) = \lim_{z\to p}\lim_{w\to q}\big(g_{\mathrm{hyp}}(z,w) - \phi(z) - \phi(w)\big).$$

From Proposition 2.4.1, for a fixed $w \in X$ with $z \in X$ approaching the parabolic fixed point $p \in \mathcal{P}$, we have

$$\begin{aligned} g_{\mathrm{hyp}}(z,w) = & \, 4\pi\kappa_p(w) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi \log y_p}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \\ & \log\big|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\big|^2 + O\big(e^{-2\pi(y_p - v_p)}\big), \end{aligned}$$

where $v_p = \mathrm{Im}(\sigma_p^{-1}w)$. Combining the above equation with Corollary 7.1.13, we find

$$\begin{aligned} \lim_{z\to p}\big(g_{\mathrm{hyp}}(z,w) - \phi(z)\big) = 4\pi\kappa_p(w) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{\alpha_p}{8\pi g} - \frac{2\pi k_{p,p}(0)}{g} + \\ \frac{C_{\mathrm{hyp}}}{8g^2} + \frac{2\pi c_X}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)} + \lim_{y_p\to\infty}\sum_{s\in\mathcal{P}}\int_{1/y_p}^{\infty}\frac{\log\zeta}{2g\,\mathrm{vol}_{\mathrm{hyp}}(X)}\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},s}(\sigma_s\xi)\frac{d\zeta}{\zeta^2}, \end{aligned} \tag{7.47}$$

where $\zeta = \mathrm{Im}(\xi)$. As $w \in X$ approaches the parabolic fixed point $q \in \mathcal{P}$ with $q \neq p$, from the Fourier expansion of Kronecker's limit function $\kappa_p(w)$, stated in Theorem 1.5.3, we have

$$4\pi\kappa_p(w) = 4\pi k_{p,q}(0) - \frac{4\pi \log v_q}{\mathrm{vol}_{\mathrm{hyp}}(X)} + O\big(e^{-2\pi v_q}\big).$$

So using Corollary 7.1.13 one more time, and substituting the above asymptotic relation into equation (7.47), we compute the limit

$$\begin{aligned} & \lim_{z\to p}\lim_{w\to q}\big(g_{\mathrm{hyp}}(z,w) - \phi(z) - \phi(w)\big) = \\ & 4\pi k_{p,q}(0) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{\alpha_p}{8\pi g} - \frac{2\pi k_{p,p}(0)}{g} - \frac{\alpha_q}{8\pi g} - \frac{2\pi k_{q,q}(0)}{g} + \\ & \frac{C_{\mathrm{hyp}}}{4g^2} + \frac{4\pi c_X}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)} + \lim_{y_p\to\infty}\sum_{s\in\mathcal{P}}\int_{1/y_p}^{\infty}\frac{\log\zeta}{2g\,\mathrm{vol}_{\mathrm{hyp}}(X)}\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},s}(\sigma_s\xi)\frac{d\zeta}{\zeta^2} + \\ & \lim_{v_q\to\infty}\sum_{s\in\mathcal{P}}\int_{1/v_q}^{\infty}\frac{\log\zeta}{2g\,\mathrm{vol}_{\mathrm{hyp}}(X)}\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},s}(\sigma_s\xi)\frac{d\zeta}{\zeta^2}. \end{aligned} \tag{7.48}$$

Using the definition of the constant α_p from (7.40), we find that the first six terms on the right-hand side of the above equation give

$$\begin{aligned} & 4\pi k_{p,q}(0) - \frac{1}{g}\bigg(\frac{2\pi|\mathcal{P}|}{\mathrm{vol}_{\mathrm{hyp}}(X)} - 2\pi\sum_{s\in\mathcal{P}}k_{s,p}(0) - \log(4\pi)\bigg) - \frac{2\pi k_{p,p}(0)}{g} - \\ & \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{1}{g}\bigg(\frac{2\pi|\mathcal{P}|}{\mathrm{vol}_{\mathrm{hyp}}(X)} - 2\pi\sum_{s\in\mathcal{P}}k_{s,q}(0) - \log(4\pi)\bigg) - \frac{2\pi k_{q,q}(0)}{g} = \\ & 4\pi k_{p,q}(0) + \frac{2\pi}{g}\bigg(\sum_{\substack{s\in\mathcal{P}\\ s\neq p}}k_{s,p}(0) + \sum_{\substack{s\in\mathcal{P}\\ s\neq q}}k_{s,q}(0)\bigg) - \frac{4\pi(|\mathcal{P}| + g)}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{2\log(4\pi)}{g}. \end{aligned}$$

Furthermore, the expression on the right-hand side of the above equation can be bounded by

$$4\pi\big|k_{p,q}(0)\big| + \frac{2\pi}{g}\bigg(\sum_{\substack{s\in\mathcal{P}\\ s\neq p}}\big|k_{s,p}(0)\big| + \sum_{\substack{s\in\mathcal{P}\\ s\neq q}}\big|k_{s,q}(0)\big|\bigg) + \frac{13|\mathcal{P}|+4\pi g}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{2\log(4\pi)}{g}. \tag{7.49}$$

Using Proposition 6.1.9, we derive the upper bound for the next two terms on the right-hand side of equation (7.48)

$$\frac{C_{\mathrm{hyp}}}{4g^2} + \frac{4\pi c_X}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)} \leq \frac{4\pi\,(d_X+1)^2}{\lambda_1\,\mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{\big|4\pi c_X\big|}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)}. \tag{7.50}$$

From Lemma 7.1.9, we have the upper bound for the absolute value of the last two terms on the right-hand side of equation (7.48)

$$\begin{aligned}
&\lim_{y_p\to\infty}\sum_{s\in\mathcal{P}}\int_{1/y_p}^{\infty}\left|\frac{\log\zeta}{2g\,\mathrm{vol}_{\mathrm{hyp}}(X)}\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},s}(\sigma_s\xi)\right|\frac{d\zeta}{\zeta^2}+\\
&\lim_{v_q\to\infty}\sum_{s\in\mathcal{P}}\int_{1/v_q}^{\infty}\left|\frac{\log\zeta}{2g\,\mathrm{vol}_{\mathrm{hyp}}(X)}\,\Delta_{\mathrm{hyp}}\,P_{\mathrm{gen},s}(\sigma_s\xi)\right|\frac{d\zeta}{\zeta^2}\leq\\
&\frac{2|\mathcal{P}|}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)}\left(1+\frac{4\pi^2}{3}\right)\leq\frac{30|\mathcal{P}|}{g\,\mathrm{vol}_{\mathrm{hyp}}(X)}.
\end{aligned} \tag{7.51}$$

The proof of the theorem follows from combining the estimates obtained in equations (7.49), (7.50), and (7.51). □

7.2 Bounds for the canonical Green's function in covers

In this section, we investigate the bounds obtained in Chapters 5 and 6 for certain sequences of Riemann surfaces similar to the study conducted in Section 5 of [11].

We start with the definition of an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume.

Definition 7.2.1. Let $\{X_N\}_{N\in\mathcal{N}}$ be a set of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume of genus $g_N\geq 1$ without torsion points indexed by $N\in\mathcal{N}\subseteq\mathbb{N}$. We say that the sequence is *admissible* if it is one of the following two types:

(1) If $\mathcal{N}=\mathbb{N}$ and $N\in\mathcal{N}$, then X_{N+1} is a finite degree cover of X_N.

(2) For $N\in\mathbb{N}_{>0}$, let

$$\begin{aligned}
Y_0(N)&=\Gamma_0(N)\backslash\mathbb{H},\\
Y_1(N)&=\Gamma_1(N)\backslash\mathbb{H},\\
Y(N)&=\Gamma(N)\backslash\mathbb{H}
\end{aligned}$$

with the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$, respectively. In each of the three cases above, let $\mathcal{N} \subseteq \mathbb{N}$ be such that $Y_0(N)$, $Y_1(N)$, $Y(N)$ has genus bigger than zero for $N \in \mathcal{N}$, respectively. We then consider here the families $\{X_N\}_{N\in\mathcal{N}}$ given by

$$\{Y_0(N)\}_{N\in\mathcal{N}},\ \ \{Y_1(N)\}_{N\in\mathcal{N}},\ \ \{Y(N)\}_{N\in\mathcal{N}}.$$

Denote by $q_{\mathcal{N}} \in \mathcal{N}$ the minimal element of the indexing set $\mathcal{N}$; in Case (1) $q_{\mathcal{N}} = 0$ and in Case (2) $q_{\mathcal{N}}$ is the smallest prime in $\mathcal{N}$.

Remark 7.2.2. It is to be noted that the family of hyperbolic modular curves do not form a single tower of hyperbolic Riemann surfaces, hence, the distinction in the above definition. However, they form a different structure which we call a net. This concept has been introduced in Section 5 of [12]. We refer the reader to Appendix A.3 for the details.

Definition 7.2.3. For each $N \in \mathcal{N}$, the non-compact hyperbolic Riemann surface of finite hyperbolic volume X_N can be realized as the quotient space $\Gamma_N\backslash\mathbb{H}$, where $\Gamma_N \subset \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian subgroup of the first kind acting by fractional linear transformations on $\mathbb{H}$. Furthermore, Γ_N admits finitely many parabolic fixed points. We denote the set of parabolic fixed points of Γ_N by $\mathcal{P}_N$ and its cardinality by $|\mathcal{P}_N|$.

Notations 7.2.4. To emphasize the dependence on X_N, for each $N \in \mathcal{N}$, we use the following notation for the rest of this section.

(1) As in (1.5), we put

$$d_{X_N} = \sup_{z\in X_N} \frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)}.$$

(2) Let $\{\lambda_{N,n}\}$ denote the set of discrete eigenvalues of the hyperbolic Laplacian Δ_{hyp} on X_N with associated orthonormal eigenfunctions $\{\varphi_{N,n}(z)\}$. As before, we assume that the eigenfunctions $\varphi_{N,n}(z)$ are real-valued.
Furthermore, let $\lambda_{N,1}$ denote the first non-zero eigenvalue of the hyperbolic Laplacian Δ_{hyp} on X_N.

(3) The canonical Green's function and the hyperbolic Green's function defined on $X_N \times X_N$ are denoted by $g_{N,\mathrm{can}}(z,w)$ and $g_{N,\mathrm{hyp}}(z,w)$, respectively.

(4) As in Section 4.2, for $z \in X_N$ and $p \in \mathcal{P}_N$, we put

$$P_N(z) = \sum_{\substack{\gamma\in\Gamma_N\setminus\{\mathrm{id}\}\\ \gamma\ \mathrm{parabolic}}} g_{\mathbb{H}}(z,\gamma z) = \sum_{p\in\mathcal{P}_N}\ \sum_{\eta\in\Gamma_{N,p}\backslash\Gamma_N} P_{N,\mathrm{gen},p}(\eta z),$$

where $P_{N,\mathrm{gen},p}(z) = \sum_{n\neq 0} g_{\mathbb{H}}(z,\gamma_p^n z)$ with γ_p a generator of the stabilizer subgroup $\Gamma_{N,p}$ of the parabolic fixed point $p \in \mathcal{P}_N$.

(5) As in (4.19) and (4.21), we define

$$C'_{N,\mathrm{par}} = \max_{z\in X_N}\Bigg(\sum_{p\in\mathcal{P}_N}\sum_{\substack{\eta\in\Gamma_{N,p}\backslash\Gamma_N\\ \eta\neq\mathrm{id}}} P_{N,\mathrm{gen},p}(\eta z)\Bigg),$$
$$C''_{N,\mathrm{par}} = \max_{z\in X_N}\big|\,\Delta_{\mathrm{hyp}}\,P_N(z)\big|.$$

(6) We denote the constant defined in (4.1) related to the Selberg zeta function associated to the Riemann surface X_N by c_{X_N}.

(7) Let $0<\varepsilon<1$ be any number such that for all $N\in\mathcal{N}$, the following condition holds true:

$$U_{N,\varepsilon}(p)\cap U_{N,\varepsilon}(q)=\emptyset \tag{7.52}$$

for all parabolic fixed points $p,q\in\mathcal{P}_N$ and $p\neq q$, where $U_{N,\varepsilon}(p)$, $U_{N,\varepsilon}(q)$ denote open coordinate disks of radius ε around $p,q\in\mathcal{P}_N$, respectively.
For a fixed $0<\varepsilon<1$ satisfying (7.52), put

$$Y_{N,\varepsilon}=X_N\backslash\bigcup_{p\in\mathcal{P}_N}U_{N,\varepsilon}(p).$$

Let $\mathcal{F}_N\subset\mathbb{H}$ denote a fixed fundamental domain of the Riemann surface X_N. Furthermore, let $\Pi_N:\mathbb{H}\longrightarrow\Gamma_N\backslash\mathbb{H}=X_N$ be the universal covering map, and define

$$Y'_{N,\varepsilon}=\Pi_N^{-1}(Y_{N,\varepsilon})\cap\mathcal{F}_N,\quad U'_{N,\varepsilon}=\Pi_N^{-1}(U_{N,\varepsilon})\cap\mathcal{F}_N.$$

(8) Let $K_{N,\mathrm{hyp}}(t;z,w)$ denote the hyperbolic heat kernel on $\mathbb{R}_{>0}\times X_N\times X_N$. As in (5.6), we define

$$C^{HK}_{N,\varepsilon}=\max_{z\in Y_{N,\varepsilon}}\big(K_{N,\mathrm{hyp}}(t_0;z)\big),$$

where t_0 is a fixed number satisfying $0<t_0<1$.

(9) As in (5.10), for $\delta>0$ and $z,w\in X_N$, we put

$$S_{\Gamma_N}(\delta;z,w)=\{\gamma\in\Gamma_N|\,d_{\mathbb{H}}(z,\gamma w)<\delta\}.$$

(10) As in (5.2), we put

$$r_{N,\varepsilon}=\inf\big\{d_{\mathbb{H}}(z,\gamma z)|\,\gamma\in\Gamma_N\backslash\{\mathrm{id}\},\,z\in Y'_{N,\varepsilon}\big\}.$$

(11) As in (5.15), we put

$$c_{N,\varepsilon}=\inf\big\{d_{\mathbb{H}}(z,\gamma w)|\,\gamma\in\Gamma_N,\,z\in Y'_{N,\varepsilon},\,w\in\partial Y'_{N,\varepsilon/2}\big\}.$$

(12) As in (6.31), we put

$$\widetilde{r}_{N,\varepsilon}=\min\{r_{N,\varepsilon/2},c_{N,\varepsilon}\}.$$

(13) The constants $B_{N,\varepsilon,\alpha,\delta}$ and $C_{N,\varepsilon,\alpha,\delta}$ associated to the compact subset $Y_{N,\varepsilon}$ of X_N are as defined in Theorems 5.2.11 and 6.1.11, respectively.

In the following five lemmas, we obtain bounds for the terms that contribute to the constant $C_{N,\varepsilon,\alpha,\delta}$ through covers and for families of modular curves. We study the behavior of the quantities d_{X_N}, c_{X_N}, $C'_{N,\mathrm{par}}$, $C''_{N,\mathrm{par}}$, $C^{HK}_{N,\varepsilon}$, and $r_{N,\varepsilon}$ for $N \in \mathcal{N}$ sufficiently large. We will employ the same techniques as the ones used in the proof of Lemma 5.3 of [11].

Lemma 7.2.5. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume. Then, with the notations as in 7.2.4, we have the following upper bounds:*

(1) For any $N \in \mathcal{N}$, we have

$$d_{X_N} = O_{q_{\mathcal{N}}}(1).$$

(2) For any $N \in \mathcal{N}$, we have

$$c_{X_N} = O_{q_{\mathcal{N}}}\left(\frac{g_N}{\lambda_{N,1}}\right).$$

Proof. The proof of the lemma follows from Lemma 5.3 of [11]. □

Notation 7.2.6. For $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ a Fuchsian subgroup of the first kind, let $\mathcal{M}_{\mathrm{par}}(\Gamma)$ denote the set of maximal parabolic subgroups of Γ. Note that for $P \in \mathcal{M}_{\mathrm{par}}(\Gamma)$, we have $P = \langle\gamma_P\rangle \in \mathcal{M}_{\mathrm{par}}(\Gamma)$, where γ_P denotes a generator of the maximal parabolic subgroup P. Furthermore, there exists a scaling matrix σ_P satisfying the condition

$$\sigma_P^{-1}\gamma_P\sigma_P = \gamma_\infty, \text{ where } \gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{7.53}$$

Remark 7.2.7. Let Γ be a subgroup of finite index in $\Gamma_0 \subset \mathrm{PSL}_2(\mathbb{R})$, a Fuchsian subgroup of the first kind. Then, there is a bijection

$$\varphi : \mathcal{M}_{\mathrm{par}}(\Gamma) \longrightarrow \mathcal{M}_{\mathrm{par}}(\Gamma_0),$$

which is given as follows. For each $P \in \mathcal{M}_{\mathrm{par}}(\Gamma)$, there exists a maximal parabolic subgroup $P_0 \subset \Gamma_0$ containing P, and we set $\varphi(P) = P_0$; the inverse map is given by $\varphi^{-1}(P_0) = P_0 \cap \Gamma$.

Furthermore, the scaling matrices σ_{P_0} and σ_P of the parabolic subgroups P_0 and P, respectively, can be chosen such that they satisfy the relation

$$\sigma_{P_0} = \sigma_P \begin{pmatrix} 1/\sqrt{n_{P_0P}} & 0 \\ 0 & \sqrt{n_{P_0P}} \end{pmatrix}, \tag{7.54}$$

where $n_{P_0P} = [P_0 : P]$.

Lemma 7.2.8. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume. Then, with the notations as in 7.2.4, for any $N \in \mathcal{N}$, we have*

$$C'_{N,\mathrm{par}} = O_{q_{\mathcal{N}}}(1).$$

Proof. We first prove the lemma for $\{X_N\}_{N\in\mathcal{N}}$, an admissible sequence of Riemann surfaces of type (1). In order to do so, we need to consider the pair of Riemann surfaces X_N and $X_{q_\mathcal{N}}$, where X_N is a finite degree cover of $X_{q_\mathcal{N}}$.

For any $N\in\mathcal{N}$ and $X_N=\Gamma_N\backslash\mathbb{H}$, consider the set

$$\mathcal{P}(\Gamma_N)=\big\{\Gamma_{N,p}\,|\,p\in\mathcal{P}_N\big\},$$

where $\Gamma_{N,p}$ denotes the stabilizer subgroup of the parabolic fixed point $p\in\mathcal{P}_N$. Keeping in mind that the set $\mathcal{P}_N$ is in bijection with the set of conjugacy classes of maximal parabolic subgroups of Γ_N, we have the equality

$$\bigcup_{p\in\mathcal{P}_N}\ \bigcup_{\eta\in\Gamma_{N,p}\backslash\Gamma_N}\eta^{-1}\Gamma_{N,p}\eta=\bigcup_{P\in\mathcal{M}_{\mathrm{par}}(\Gamma_N)}P\,,$$

from which we can conclude that

$$\sum_{p\in\mathcal{P}_N}\ \sum_{\substack{\eta\in\Gamma_{N,p}\backslash\Gamma_N\\ \eta\neq\mathrm{id}}}P_{N,\mathrm{gen},p}(\eta z)=\sum_{\substack{P\in\mathcal{M}_{\mathrm{par}}(\Gamma_N)\\ P\notin\mathcal{P}(\Gamma_N)}}\ \sum_{n\neq 0}g_{\mathbb{H}}(z,\gamma_P^n z),$$

where γ_P denotes a generator of $P\in\mathcal{M}_{\mathrm{par}}(\Gamma_N)$. For $P\in\mathcal{M}_{\mathrm{par}}(\Gamma_N)$, from the definition of the scaling matrix σ_P as defined in (7.53), we find

$$\begin{aligned}\sum_{p\in\mathcal{P}_N}\ \sum_{\substack{\eta\in\Gamma_{N,p}\backslash\Gamma_N\\ \eta\neq\mathrm{id}}}P_{N,\mathrm{gen},p}(\eta z)&=\sum_{\substack{P\in\mathcal{M}_{\mathrm{par}}(\Gamma_N)\\ P\notin\mathcal{P}(\Gamma_N)}}\ \sum_{n\neq 0}g_{\mathbb{H}}(\sigma_P^{-1}z,\gamma_\infty^n\sigma_P^{-1}z)\\ &=\sum_{\substack{P\in\mathcal{M}_{\mathrm{par}}(\Gamma_N)\\ P\notin\mathcal{P}(\Gamma_N)}}\ \sum_{n\neq 0}\log\left|\frac{4y_P^2+n^2}{n^2}\right|,\end{aligned}\tag{7.55}$$

where $y_P=\mathrm{Im}(\sigma_P^{-1}z)$. From Remark 7.2.7, we have a bijective map

$$\varphi_{N,q_\mathcal{N}}:\mathcal{M}_{\mathrm{par}}\big(\Gamma_N\big)\longrightarrow\mathcal{M}_{\mathrm{par}}\big(\Gamma_{q_\mathcal{N}}\big),$$

sending $P\in\mathcal{M}_{\mathrm{par}}(\Gamma_N)$ to $P_0=\varphi_{N,q_\mathcal{N}}(P)\in\mathcal{M}_{\mathrm{par}}(\Gamma_{q_\mathcal{N}})$. Then, for $z\in\mathbb{H}$, using the relation stated in equation (7.54), we have

$$y_P=\mathrm{Im}(\sigma_P^{-1}z)=\begin{pmatrix}1/\sqrt{n_{P_0P}} & 0\\ 0 & \sqrt{n_{P_0P}}\end{pmatrix}\mathrm{Im}(\sigma_{P_0}^{-1}z)=\frac{y_{P_0}}{n_{P_0P}},\tag{7.56}$$

where $n_{P_0P}=[P_0:P]$. For $z\in\mathbb{H}$, using the above relation and the bijection between the sets $\mathcal{M}_{\mathrm{par}}(\Gamma_N)$ and $\mathcal{M}_{\mathrm{par}}(\Gamma_{q_\mathcal{N}})$, we derive

$$\begin{aligned}\sum_{\substack{P\in\mathcal{M}_{\mathrm{par}}(\Gamma_N)\\ P\notin\mathcal{P}(\Gamma_N)}}\ \sum_{n\neq 0}\log\left|\frac{4y_P^2+n^2}{n^2}\right|&\leq\sum_{\substack{P_0\in\mathcal{M}_{\mathrm{par}}(\Gamma_{q_\mathcal{N}})\\ P_0\notin\mathcal{P}(\Gamma_{q_\mathcal{N}})}}\ \sum_{n\neq 0}\log\left|\frac{4y_{P_0}^2/n_{P_0P}^2+n^2}{n^2}\right|\\ &\leq\sum_{\substack{P_0\in\mathcal{M}_{\mathrm{par}}(\Gamma_{q_\mathcal{N}})\\ P_0\notin\mathcal{P}(\Gamma_{q_\mathcal{N}})}}\ \sum_{n\neq 0}\log\left|\frac{4y_{P_0}^2+n^2}{n^2}\right|.\end{aligned}\tag{7.57}$$

Combining equation (7.55) with (7.57), we deduce that

$$C'_{N,\mathrm{par}} \leq C'_{q_{\mathcal{N}},\mathrm{par}} = O_{q_{\mathcal{N}}}(1),$$

which proves the lemma for the case of an admissible sequence of Riemann surfaces of type (1).

We now prove the lemma for $\{X_N\}_{N\in\mathcal{N}}$, an admissible sequence of Riemann surfaces of type (2). We prove the lemma only for the sequence of modular curves $\{Y_0(N)\}_{N\in\mathcal{N}}$, as the proof extends with notational changes to the other sequences of modular curves $\{Y_1(N)\}_{N\in\mathcal{N}}$ and $\{Y(N)\}_{N\in\mathcal{N}}$.

For any $N \in \mathcal{N}$ the modular curve $Y_0(N)$ is a finite degree cover of $Y_0(1) = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. Extending the notations from 7.2.4 to the modular curve $Y_0(1)$ and adapting the arguments from the proof for admissible sequences of Riemann surfaces of type (1), for $N \in \mathcal{N}$ sufficiently large, we have

$$C'_{N,\mathrm{par}} = O(1),$$

which trivially implies that

$$C'_{N,\mathrm{par}} = O_{q_{\mathcal{N}}}(1).$$

This completes the proof of the lemma. □

Lemma 7.2.9. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume. Then, with the notations as in 7.2.4, for any $N \in \mathcal{N}$, we have*

$$C''_{N,\mathrm{par}} = O_{q_{\mathcal{N}}}(1).$$

Proof. For the case of admissible sequences of Riemann surfaces of type (1), this result has been established as Proposition 5.4 in [14]. Using Proposition 5.4 from [14] and adapting the arguments from Lemma 7.2.8 will trivially prove the lemma for the case of admissible sequences of Riemann surfaces of type (2). □

Lemma 7.2.10. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume. Then, with the notations as in 7.2.4, for $N \in \mathcal{N}$ sufficiently large, we have*

$$C^{HK}_{N,\varepsilon} = O_{q_{\mathcal{N}},\varepsilon}\big(g_N\big).$$

Proof. We first prove the lemma for $\{X_N\}_{N\in\mathcal{N}}$, an admissible sequence of Riemann surfaces of type (1). In order to do so, we need to consider the pair of Riemann surfaces X_N and $X_{q_{\mathcal{N}}}$, where X_N is a finite degree cover of $X_{q_{\mathcal{N}}}$.

For any $N \in \mathcal{N}$ and any parabolic fixed point $p \in \mathcal{P}_N$, the covering map $\Pi_{N,q_{\mathcal{N}}} : X_N \longrightarrow X_{q_{\mathcal{N}}}$ takes the coordinate disk $U_\varepsilon(p)$ of radius ε around $p \in \mathcal{P}_N$ to the coordinate disk $U_{\varepsilon^n}(p_0)$ of radius ε^n around $\Pi_{N,q_{\mathcal{N}}}(p) = p_0 \in \mathcal{P}_{q_{\mathcal{N}}}$, where n denotes the index $[\Gamma_{q_{\mathcal{N}},p_0} : \Gamma_{N,p}]$. Let $n_0 = [\Gamma_{q_{\mathcal{N}}} : \Gamma_N]$. Since, $n = [\Gamma_{q_{\mathcal{N}},p_0} : \Gamma_{N,p}] \leq [\Gamma_{q_{\mathcal{N}}} : \Gamma_N] = n_0$, we have $\Pi_{N,q_{\mathcal{N}}}(Y_{N,\varepsilon}) \subseteq Y_{q_{\mathcal{N}},\varepsilon^{n_0}}$.

As Γ_N is a subgroup of $\Gamma_{q_{\mathcal{N}}}$, for all $z \in Y_{N,\varepsilon}$, it trivially follows that

$$C^{HK}_{N,\varepsilon} = \max_{z\in Y_{N,\varepsilon}} \big(K_{N,\mathrm{hyp}}(t_0; z)\big) \leq \max_{z\in \Pi_{N,q_{\mathcal{N}}}(Y_{N,\varepsilon})} \big(K_{q_{\mathcal{N}},\mathrm{hyp}}(t_0; z)\big) \leq \max_{z\in Y_{q_{\mathcal{N}},\varepsilon^{n_0}}} \big(K_{q_{\mathcal{N}},\mathrm{hyp}}(t_0; z)\big) = C^{HK}_{q_{\mathcal{N}},\varepsilon^{n_0}}. \tag{7.58}$$

We now analyze the term $C^{HK}_{q_{\mathcal{N}},\varepsilon^{n_0}}$ for sufficiently large $N \in \mathcal{N}$. Without loss of generality, let us assume that $\Gamma_{q_{\mathcal{N}}}$ has only one parabolic fixed point, say p_0 with stabilizer subgroup $\Gamma_{q_{\mathcal{N}},p_0}$. From the integral formula for $K_{\mathbb{H}}(t;\rho)$ described in (1.10), we find

$$\lim_{z\to p_0} \sum_{\gamma\in\Gamma_{q_{\mathcal{N}}}\setminus\Gamma_{q_{\mathcal{N}},p_0}} K_{\mathbb{H}}(t_0; z, \gamma z) = 0. \tag{7.59}$$

Furthermore, from Proposition 3.3.5 in [7], which is a reformulation of a result from [20], for $z \in X_{q_{\mathcal{N}}}$ approaching p_0, we have

$$\sum_{\gamma\in\Gamma_{q_{\mathcal{N}},p_0}} K_{\mathbb{H}}(t_0; z, \gamma z) = \frac{e^{-t_0/4}\cdot y_{p_0}}{\sqrt{4\pi t_0}}\bigg(1 + \sum_{n\neq 0} e^{-4\pi^2 n^2 y_{p_0}^2 t_0} + O\Big(\frac{1}{y_{p_0}^2}\Big)\bigg), \tag{7.60}$$

where y_{p_0} denotes $\mathrm{Im}(\sigma_{p_0}^{-1}z)$ with σ_{p_0} being a scaling matrix of the parabolic fixed point p_0. Hence, for $N \in \mathcal{N}$ sufficiently large, combining equations (7.59) and (7.60), we derive

$$C^{HK}_{q_{\mathcal{N}},\varepsilon^{n_0}} = -\frac{e^{-t_0/4}}{\sqrt{4\pi t_0}}\cdot\frac{n_0\log\varepsilon}{2\pi} + O_{q_{\mathcal{N}}}(1) = O_{q_{\mathcal{N}},\varepsilon}(n_0). \tag{7.61}$$

For $N \in \mathcal{N}$ sufficiently large, from the Riemann-Hurwitz formula, we have

$$n_0 = [\Gamma_{q_{\mathcal{N}}} : \Gamma_N] = O_{q_{\mathcal{N}}}(g_N). \tag{7.62}$$

Combining the estimates obtained in (7.61) and (7.62) completes the proof of the lemma for the case of an admissible sequence of Riemann surfaces of type (1).

We now prove the lemma for $\{X_N\}_{N\in\mathcal{N}}$, an admissible sequence of Riemann surfaces of type (2). We prove the lemma only for the sequence of modular curves $\{Y_0(N)\}_{N\in\mathcal{N}}$, as the proof extends with notational changes to the other sequences of modular curves $\{Y_1(N)\}_{N\in\mathcal{N}}$ and $\{Y(N)\}_{N\in\mathcal{N}}$.

For any $N \in \mathcal{N}$ the modular curve $Y_0(N)$ is a finite degree cover of $Y_0(1) = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. Extending the notations from 7.2.4 to the modular curve $Y_0(1)$ and adapting the arguments from the proof for admissible sequences of Riemann surfaces of type (1), for $N \in \mathcal{N}$ sufficiently large, we have

$$C^{HK}_{N,\varepsilon} = O_\varepsilon(g_N),$$

which trivially implies that

$$C^{HK}_{N,\varepsilon} = O_{q_{\mathcal{N}},\varepsilon}\big(g_N\big).$$

This completes the proof of the lemma. $\square$

Lemma 7.2.11. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume. Then, with the notations as in 7.2.4, for $N\in\mathcal{N}$ sufficiently large, we have*

$$\frac{1}{\sinh(r_{N,\varepsilon}/2)} = O_{q_{\mathcal{N}},\varepsilon}\big(g_N\big).$$

Proof. We first prove the lemma for $\{X_N\}_{N\in\mathcal{N}}$, an admissible sequence of Riemann surfaces of type (1). In order to do so, we need to consider the pair of Riemann surfaces X_N and $X_{q_{\mathcal{N}}}$, where X_N is a finite degree cover of $X_{q_{\mathcal{N}}}$.

For any $N\in\mathcal{N}$, the covering map $\Pi_{N,q_{\mathcal{N}}}: X_N \longrightarrow X_{q_{\mathcal{N}}}$ induces a map from $\mathcal{F}_N$ to $\mathcal{F}_{q_{\mathcal{N}}}$, the fundamental domain of X_N to the fundamental domain of $X_{q_{\mathcal{N}}}$, respectively, which we again denote by $\Pi_{N,q_{\mathcal{N}}}$. Let $n_0=[\Gamma_{q_{\mathcal{N}}}:\Gamma_N]$. From the definition of the injectivity radius, we have

$$\begin{aligned}
r_{N,\varepsilon} &= \inf\big\{d_{\mathbb{H}}(z,\gamma z)\,|\,\gamma\in\Gamma_N\backslash\{\mathrm{id}\},\, z\in Y'_{N,\varepsilon}\big\}\\
&= \inf\big\{d_{\mathbb{H}}(\gamma' z,\gamma\gamma' z)\,|\,\gamma\in\Gamma_N\backslash\{\mathrm{id}\},\,\gamma'\in\Gamma_{q_{\mathcal{N}}},\, z\in\Pi_{N,q_{\mathcal{N}}}\big(Y'_{N,\varepsilon}\big)\big\}\\
&= \inf\big\{d_{\mathbb{H}}(z,\gamma'^{-1}\gamma\gamma' z)\,|\,\gamma\in\Gamma_N\backslash\{\mathrm{id}\},\,\gamma'\in\Gamma_{q_{\mathcal{N}}},\, z\in\Pi_{N,q_{\mathcal{N}}}\big(Y'_{N,\varepsilon}\big)\big\}\\
&\geq \inf\big\{d_{\mathbb{H}}(z,\gamma'' z)\,|\,\gamma''\in\Gamma_{q_{\mathcal{N}}}\backslash\{\mathrm{id}\},\, z\in\Pi_{N,q_{\mathcal{N}}}(Y'_{N,\varepsilon})\big\}.
\end{aligned}\tag{7.63}$$

Since $\Pi_{N,q_{\mathcal{N}}}(Y_{N,\varepsilon})\subseteq Y_{q_{\mathcal{N}},\varepsilon^{n_0}}$, using the above inequality, we deduce that

$$r_{q_{\mathcal{N}},\varepsilon^{n_0}} \leq r_{N,\varepsilon}.\tag{7.64}$$

As $\sinh(t)$ is a monotone increasing function for $t\in\mathbb{R}_{\geq 0}$, we have

$$\sinh\big(r_{q_{\mathcal{N}},\varepsilon^{n_0}}/2\big)\leq \sinh(r_{N,\varepsilon}/2).$$

From Lemma 5.1.6, for $N\in\mathcal{N}$ sufficiently large, we have

$$\sinh\big(r_{q_{\mathcal{N}},\varepsilon^{n_0}}/2\big) = -\frac{\pi}{n_0\log\varepsilon}.$$

Hence, for $N\in\mathcal{N}$ sufficiently large, we have

$$\frac{1}{\sinh(r_{N,\varepsilon}/2)}\leq\frac{1}{\sinh\big(r_{q_{\mathcal{N}},\varepsilon^{n_0}}/2\big)} = -\frac{n_0\log\varepsilon}{\pi} = O_{q_{\mathcal{N}},\varepsilon}(n_0).$$

Combining the above estimate with (7.62) completes the proof of the lemma for the case of an admissible sequence of Riemann surfaces of type (1).

We now prove the lemma for $\{X_N\}_{N\in\mathcal{N}}$, an admissible sequence of Riemann surfaces of type (2). We prove the lemma only for the sequence of modular curves $\{Y_0(N)\}_{N\in\mathcal{N}}$, as the proof extends with notational changes to the other sequences of modular curves $\{Y_1(N)\}_{N\in\mathcal{N}}$ and $\{Y(N)\}_{N\in\mathcal{N}}$.

For any $N \in \mathcal{N}$ the modular curve $Y_0(N)$ is a finite degree cover of $Y_0(1) = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. Extending the notations from 7.2.4 to the modular curve $Y_0(1)$ and adapting the arguments from the proof for admissible sequences of Riemann surfaces of type (1), for $N \in \mathcal{N}$ sufficiently large, we have

$$\frac{1}{\sinh(r_{N,\varepsilon}/2)} = O_{\varepsilon}\big(g_N\big).$$

which trivially implies that

$$\frac{1}{\sinh(r_{N,\varepsilon}/2)} = O_{q_{\mathcal{N}},\varepsilon}\big(g_N\big).$$

This completes the proof of the lemma. □

The following theorem is an extension of Theorem 5.5 in [11] to admissible sequences of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume.

Theorem 7.2.12. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume and genus $g_N \geq 1$. Then, with the notations as in 7.2.4, for any $N \in \mathcal{N}$ sufficiently large, $0 < \varepsilon < 1$, $\alpha > 0$, and $\delta > 0$, we have the upper bound*

$$\sup_{z,w\in Y_{N,\varepsilon}} \Bigg| g_{N,\mathrm{hyp}}(z,w) - \sum_{n\colon 0<\lambda_{N,n}<\alpha} \frac{4\pi}{\lambda_{N,n}} \varphi_{N,n}(z)\varphi_{N,n}(w) - \sum_{\gamma\in S_{\Gamma_N}(\delta;z,w)} g_{\mathbb{H}}(z,w) \Bigg| = O_{q_{\mathcal{N}},\varepsilon,\alpha,\delta}\big(g_N^2\big).$$

Proof. From Theorem 5.2.11, for any $N \in \mathcal{N}$, $0 < \varepsilon < 1$, $\alpha > 0$, and $\delta > 0$, we have

$$\sup_{z,w\in Y_{N,\varepsilon}} \Bigg| g_{N,\mathrm{hyp}}(z,w) - \sum_{n\colon 0<\lambda_{N,n}<\alpha} \frac{4\pi}{\lambda_{N,n}} \varphi_{N,n}(z)\varphi_{N,n}(w) - \sum_{\gamma\in S_{\Gamma_N}(\delta;z,w)} g_{\mathbb{H}}(z,w) \Bigg| \leq B_{N,\varepsilon,\alpha,\delta}.$$

Recall that in (5.9) $\delta_{N,\varepsilon}$ is defined as a fixed number satisfying

$$\delta_{N,\varepsilon} > \max\{\delta_0,\, 4r_{N,\varepsilon} + 5\},$$

where δ_0 is a constant, which is as defined in Lemma 5.2.1. We first assume that $\delta > \delta_{N,\varepsilon}$ and derive an upper bound for $B_{N,\varepsilon,\alpha,\delta}$. From Theorem 5.2.11, for $\delta > \delta_{N,\varepsilon}$, we have

$$B_{N,\varepsilon,\alpha,\delta} = 4\pi\bigg(C_{N,\varepsilon}^{HK}e^{\alpha t_0} + \frac{c_0\sinh(r_{N,\varepsilon})\sinh(\delta)}{8\delta^2\sinh^2(r_{N,\varepsilon}/2)} + \frac{c_0 e^{2r_{N,\varepsilon}}}{2\pi\sinh^2(r_{N,\varepsilon}/2)} \\ + \frac{4c_\infty\sinh(\delta + r_{N,\varepsilon})}{\sinh(r_{N,\varepsilon})} + \frac{C_{N,\varepsilon}^{HK}}{\alpha}\bigg).$$

We now estimate each of the terms which contribute to $B_{N,\varepsilon,\alpha,\delta}$.

From Lemma 7.2.10, for $N \in \mathcal{N}$ sufficiently large, we have the following estimates of the first and the last term of $B_{N,\varepsilon,\alpha,\delta}$ (note that $0 < t_0 < 1$ is fixed as in Definition 5.1.7):

$$C_{N,\varepsilon}^{HK}e^{\alpha t_0} = O_{q_{\mathcal{N}},\varepsilon,\alpha}\big(g_N\big) \quad \text{and} \quad \frac{C_{N,\varepsilon}^{HK}}{\alpha} = O_{q_{\mathcal{N}},\varepsilon,\alpha}\big(g_N\big). \tag{7.65}$$

In order to proceed, we need an upper bound for $r_{N,\varepsilon}$. For any $N \in \mathcal{N}$ and a parabolic fixed point $p \in \mathcal{P}_N$, observe that

$$\begin{aligned} r_{N,\varepsilon} &= \inf\big\{d_{\mathbb{H}}(z,\gamma z)\big|\,\gamma \in \Gamma_N\backslash\{\mathrm{id}\},\, z \in Y'_{N,\varepsilon}\big\} \\ &\leq \inf\big\{d_{\mathbb{H}}(z,\gamma z)\big|\,\gamma \in \Gamma_{N,p}\backslash\{\mathrm{id}\},\, z \in \partial U'_{N,\varepsilon}(p)\big\}, \end{aligned} \tag{7.66}$$

where $\Gamma_{N,p}$ denotes the stabilizer subgroup of the parabolic fixed point p. Recall that from equation (5.3), we have

$$\cosh(d_{\mathbb{H}}(z,w)) = 1 + 2u(z,w), \text{ where } u(z,w) = \frac{|z-w|^2}{4\operatorname{Im}(z)\operatorname{Im}(w)}.$$

Since $\cosh(t)$ is a positive monotone increasing function for $t \in \mathbb{R}_{\geq 0}$, from inequality (7.66), we have

$$\cosh(r_{N,\varepsilon}) \leq \inf_{\substack{\gamma\in\Gamma_{N,p}\backslash\{\mathrm{id}\}\\ z\in\partial U'_{N,\varepsilon}(p)}} \cosh(d_{\mathbb{H}}(z,\gamma z)) = 1 + 2\inf_{\substack{\gamma\in\Gamma_{N,p}\backslash\{\mathrm{id}\}\\ z\in\partial U'_{N,\varepsilon}(p)}} u(z,\gamma z). \tag{7.67}$$

From the computation carried out in equation (5.5), we derive

$$\inf_{\substack{\gamma\in\Gamma_{N,p}\backslash\{\mathrm{id}\}\\ z\in\partial U'_{N,\varepsilon}(p)}} u(z,\gamma z) = \frac{\pi^2}{(\log\varepsilon)^2}.$$

Combining the above equation with inequality (7.67), we arrive at the upper bound

$$\cosh(r_{N,\varepsilon}) = O_\varepsilon(1). \tag{7.68}$$

Combining (7.68) with Lemma 7.2.11, for $N \in \mathcal{N}$ sufficiently large, we have the following estimates of the second, third, and fourth term of $B_{N,\varepsilon,\alpha,\delta}$:

- $$\frac{c_0 \sinh(r_{N,\varepsilon}) \sinh(\delta)}{8\delta^2 \sinh^2(r_{N,\varepsilon}/2)} = \frac{c_0 \cosh(r_{N,\varepsilon}/2) \sinh(\delta)}{4\delta^2 \sinh(r_{N,\varepsilon}/2)} =$$
  $$O_{\varepsilon,\delta}\bigg(\frac{1}{\sinh(r_{N,\varepsilon}/2)}\bigg) = O_{q_{\mathcal{N}},\varepsilon,\delta}\big(g_N\big),$$
- $$\frac{c_0 e^{2r_{N,\varepsilon}}}{2\pi \sinh^2(r_{N,\varepsilon}/2)} \leq \frac{c_0 \cosh(2r_{N,\varepsilon})}{\pi \sinh^2(r_{N,\varepsilon}/2)} =$$
  $$O_{\varepsilon}\bigg(\frac{1}{\sinh^2(r_{N,\varepsilon}/2)}\bigg) = O_{q_{\mathcal{N}},\varepsilon}\big(g_N^2\big),$$
- $$\frac{4c_\infty \sinh(\delta + r_{N,\varepsilon})}{\sinh(r_{N,\varepsilon})} = 4c_\infty \cosh(\delta) + \frac{4c_\infty \sinh(\delta) \cosh(r_{N,\varepsilon})}{\sinh(r_{N,\varepsilon})} =$$
  $$O_{\varepsilon,\delta}\bigg(\frac{1}{\sinh(r_{N,\varepsilon})}\bigg) = O_{q_{\mathcal{N}},\varepsilon,\delta}\big(g_N\big). \tag{7.69}$$

For $N \in \mathcal{N}$ sufficiently large and $\delta > \delta_{N,\varepsilon}$, combining the above estimates with (7.65), we arrive at the upper bound

$$B_{N,\varepsilon,\alpha,\delta} = O_{q_{\mathcal{N}},\varepsilon,\alpha,\delta}\big(g_N^2\big).$$

We now compute an upper bound for $B_{N,\varepsilon,\alpha,\delta}$, when $\delta \leq \delta_{N,\varepsilon}$. From Theorem 5.2.11, for $\delta \leq \delta_{N,\varepsilon}$, we have

$$B_{N,\varepsilon,\alpha,\delta} = B_{N,\varepsilon,\alpha,\delta_{N,\varepsilon}} + \frac{\sinh(\delta_{N,\varepsilon} + r_{N,\varepsilon})}{\sinh(r_{N,\varepsilon})} \big|\log\big(\tanh^2(\delta/2)\big)\big|. \tag{7.70}$$

Choose $\delta_{N,\varepsilon} = \max\{\delta_0 + 1, 4r_{N,\varepsilon} + 6\}$. From the choice of $\delta_{N,\varepsilon}$, we have $\delta_{N,\varepsilon} \geq 6$. Furthermore, from (7.68), we deduce the upper bound for $\delta_{N,\varepsilon}$

$$\delta_{N,\varepsilon} = O_{\varepsilon}(1). \tag{7.71}$$

Hence, for $N \in \mathcal{N}$ sufficiently large, substituting $\delta = \delta_{N,\varepsilon}$ in (7.69), we have the estimate for the first term on the right-hand side of equation (7.70)

$$B_{N,\varepsilon,\alpha,\delta_{N,\varepsilon}} = O_{q_{\mathcal{N}},\varepsilon,\alpha}\big(g_N^2\big). \tag{7.72}$$

Combining (7.71) with Lemma 7.2.11, for $N \in \mathcal{N}$ sufficiently large, we have the estimate for the remaining term on the right-hand side of equation (7.70)

$$\frac{\sinh(\delta_{N,\varepsilon} + r_{N,\varepsilon})}{\sinh(r_{N,\varepsilon})} \big|\log\big(\tanh^2(\delta/2)\big)\big| \leq \frac{\sinh(2\delta_{N,\varepsilon})}{\sinh(r_{N,\varepsilon})} \big|\log\big(\tanh^2(\delta/2)\big)\big| =$$
$$O_{\varepsilon,\delta}\bigg(\frac{1}{\sinh(r_{N,\varepsilon})}\bigg) = O_{q_{\mathcal{N}},\varepsilon,\delta}\big(g_N\big). \tag{7.73}$$

Hence, for $N \in \mathcal{N}$ sufficiently large and $\delta \leq \delta_{N,\varepsilon}$, combining the estimates obtained in (7.72) and (7.73), we arrive at the upper bound

$$B_{N,\varepsilon,\alpha,\delta} = O_{q_{\mathcal{N}},\varepsilon,\alpha,\delta}\big(g_N^2\big),$$

which completes the proof of the theorem. $\square$

The following theorem is an extension of Theorem 5.6 in [11] to admissible sequences of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume.

Theorem 7.2.13. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume and genus $g_N \geq 1$. Then, with the notations as in 7.2.4, for any $N \in \mathcal{N}$ sufficiently large, $0 < \varepsilon < 1$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have the upper bound*

$$\sup_{z,w\in Y_{N,\varepsilon}} \big|g_{N,\mathrm{hyp}}(z,w) - g_{N,\mathrm{can}}(z,w)\big| = O_{q_{\mathcal{N}},\varepsilon,\delta}\bigg(g_N\,|\mathcal{P}_N|\bigg(1+\frac{1}{\lambda_{N,1}}\bigg)\bigg).$$

Proof. From Theorem 6.1.12, for any $N \in \mathcal{N}$, $0 < \varepsilon < 1$, $\alpha \in (0, \lambda_{N,1})$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have

$$\sup_{z,w\in Y_{N,\varepsilon}} \big|g_{N,\mathrm{hyp}}(z,w) - g_{N,\mathrm{can}}(z,w)\big| \leq 2\,C_{N,\varepsilon,\alpha,\delta},$$

where

$$\begin{aligned} C_{N,\varepsilon,\alpha,\delta} &= \frac{B_{N,\varepsilon/2,\alpha,\delta}}{2g_N}\bigg(1+|\mathcal{P}_N| - \frac{|\mathcal{P}_N|\,C''_{N,\mathrm{par}}}{\log(\varepsilon/2)}\bigg) - \frac{2\,|\mathcal{P}_N|\log(\varepsilon/2)}{g_N} + \frac{C'_{N,\mathrm{par}}}{2g_N} + \\ &\quad \frac{2\pi\big(d_{X_N}+1\big)^2}{\lambda_{N,1}\,\mathrm{vol}_{\mathrm{hyp}}(X_N)} + \frac{2\pi|c_{X_N}-1|}{g_N\,\mathrm{vol}_{\mathrm{hyp}}(X_N)} - \frac{C''_{N,\mathrm{par}}}{g_N\log(\varepsilon/2)}. \end{aligned} \tag{7.74}$$

We choose $\alpha = \lambda_{N,1}/2$ and derive estimates of each of the terms which contribute to $C_{N,\varepsilon,\alpha,\delta}$.

We start with estimating the term $B_{N,\varepsilon/2,\alpha,\delta}$ for $\alpha = \lambda_{N,1}/2$. For $N \in \mathcal{N}$ sufficiently large, substituting $\alpha = \lambda_{N,1}/2$ in (7.65) and (7.69), from the proof of Theorem 7.2.12, we have the estimate

$$\frac{B_{N,\varepsilon/2,\,\lambda_{N,1}/2,\,\delta}}{g_N} = O_{q_{\mathcal{N}},\varepsilon,\delta}\bigg(g_N\bigg(1+\frac{1}{\lambda_{N,1}}\bigg)\bigg). \tag{7.75}$$

From Lemmas 7.2.8 and 7.2.9, we know that the constants $C'_{N,\mathrm{par}}$ and $C''_{N,\mathrm{par}}$ are bounded by constants which depend only on the Riemann surface $X_{q_{\mathcal{N}}}$, respectively. Using (7.75), for $N \in \mathcal{N}$ sufficiently large and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have the following estimate of the first line on the right-hand side of equation (7.74)

$$\begin{aligned} &\frac{B_{N,\varepsilon/2,\,\lambda_{N,1}/2,\,\delta}}{2g_N}\bigg(1+|\mathcal{P}_N| - \frac{|\mathcal{P}_N|\,C''_{N,\mathrm{par}}}{\log(\varepsilon/2)}\bigg) - \frac{2\,|\mathcal{P}_N|\log(\varepsilon/2)}{g_N} + \frac{C'_{N,\mathrm{par}}}{2g_N} = \\ &O_{q_{\mathcal{N}},\varepsilon,\delta}\bigg(g_N\,|\mathcal{P}_N|\bigg(1+\frac{1}{\lambda_{N,1}}\bigg)\bigg). \end{aligned} \tag{7.76}$$

From Lemmas 7.2.5 and 7.2.9, we know that the constants d_{X_N} and $C''_{N,\mathrm{par}}$ are bounded by constants which depend only on the Riemann surface $X_{q_{\mathcal{N}}}$, respectively. Furthermore, from Lemma 7.2.5 and the fact that $g_N \leq \mathrm{vol}_{\mathrm{hyp}}(X_N)$, we have

$$\frac{2\pi|c_{X_N}-1|}{g_N\,\mathrm{vol}_{\mathrm{hyp}}(X_N)} = O_{q_{\mathcal{N}}}\Big(\frac{1}{g_N\lambda_{N,1}}\Big).$$

So for $N \in \mathcal{N}$ sufficiently large, we derive the following estimate of the second line on the right-hand side of equation (7.74)

$$\frac{2\pi\big(d_{X_N}+1\big)^2}{\lambda_{N,1}\,\mathrm{vol}_{\mathrm{hyp}}(X_N)} + \frac{2\pi|c_{X_N}-1|}{g_N\,\mathrm{vol}_{\mathrm{hyp}}(X_N)} - \frac{C''_{N,\mathrm{par}}}{g_N\log(\varepsilon/2)} = O_{q_{\mathcal{N}},\varepsilon}\bigg(\frac{1}{g_N}\Big(1+\frac{1}{\lambda_{N,1}}\Big)\bigg). \tag{7.77}$$

Hence, combining the estimates obtained in (7.76) and (7.77) completes the proof of the theorem. □

The following theorem is an extension of Corollary 5.7 in [11] to admissible sequences of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume.

Theorem 7.2.14. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume and genus $g_N \geq 1$. Then, with the notations as in 7.2.4, for any $N \in \mathcal{N}$ sufficiently large, $0<\varepsilon<1$, and $\delta \in (0,\widetilde{r}_{N,\varepsilon})$, we have the upper bound*

$$\sup_{z,w\in Y_{N,\varepsilon}}\Big|g_{N,\mathrm{can}}(z,w) - \sum_{\gamma\in S_{\Gamma_N}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Big| = O_{q_{\mathcal{N}},\varepsilon,\delta}\bigg(g_N\,|\mathcal{P}_N|\Big(1+\frac{1}{\lambda_{N,1}}\Big)\bigg).$$

Proof. From Theorem 6.1.13, for any $N\in\mathcal{N}$, $0<\varepsilon<1$, $\alpha\in(0,\lambda_{N,1})$, and $\delta\in(0,\widetilde{r}_{N,\varepsilon})$, we have

$$\sup_{z,w\in Y_{N,\varepsilon}}\Big|g_{N,\mathrm{can}}(z,w) - \sum_{\gamma\in S_{\Gamma_N}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Big| \leq A_{N,\varepsilon,\alpha,\delta},$$

where $A_{N,\varepsilon,\alpha,\delta} = B_{N,\varepsilon,\alpha,\delta} + 2\,C_{N,\varepsilon,\alpha,\delta}$. For $N\in\mathcal{N}$ sufficiently large, $\alpha\in(0,\lambda_{N,1})$, and $\delta\in(0,\widetilde{r}_{N,\varepsilon})$, the claimed upper bound follows from the upper bounds derived for the quantities $B_{N,\varepsilon,\alpha,\delta}$ and $C_{N,\varepsilon,\alpha,\delta}$ in the proof of Theorem 7.2.13. □

The following theorem is an extension of Theorem 7.2.14 to parabolic fixed points, when one variable is in the neighborhood of a parabolic fixed point and the other remains bounded away from the parabolic fixed points.

Theorem 7.2.15. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume and genus $g_N\geq 1$, and let $p\in\mathcal{P}_N$ be a parabolic fixed point. Then, with the notations as in 7.2.4, for* *any $N \in \mathcal{N}$ sufficiently large, $0 < \varepsilon < 1$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have the upper bound*

$$\sup_{\substack{z\in U_{N,\varepsilon}(p)\\ w\in Y_{N,\varepsilon}}} \bigg| g_{N,\mathrm{can}}(z,w) - \sum_{\gamma\in S_{\Gamma_N}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\bigg| =$$
$$O_{q_{\mathcal{N}},\varepsilon,\delta}\bigg(g_N\,|\mathcal{P}_N|\bigg(1+\frac{1}{\lambda_{N,1}}\bigg)\bigg).$$

Proof. From Theorem 6.2.7, for any $N \in \mathcal{N}$, $0 < \varepsilon < 1$, $\alpha \in (0, \lambda_{N,1})$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have

$$\sup_{\substack{z\in U_{N,\varepsilon}(p)\\ w\in Y_{N,\varepsilon}}} \bigg| g_{N,\mathrm{can}}(z,w) - \sum_{\gamma\in S_{\Gamma_N}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\bigg| \leq$$
$$A_{N,\varepsilon,\alpha,\delta} + \frac{\pi d_{X_N}}{(\log\varepsilon)^2\,\mathrm{vol}_{\mathrm{hyp}}(X_N)}.$$

From the proof of Theorem 7.2.14, for $N \in \mathcal{N}$ sufficiently large, $0 < \varepsilon < 1$, $\alpha \in (0, \lambda_{N,1})$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have the following upper bound for the first term on the right-hand side of the above inequality

$$A_{N,\varepsilon,\alpha,\delta} = O_{q_{\mathcal{N}},\varepsilon,\delta}\bigg(g_N\,|\mathcal{P}_N|\bigg(1+\frac{1}{\lambda_{N,1}}\bigg)\bigg). \tag{7.78}$$

From Lemma 7.2.5, we have the following upper bound for the second term

$$\frac{\pi d_{X_N}}{(\log\varepsilon)^2\,\mathrm{vol}_{\mathrm{hyp}}(X_N)} = O_{q_{\mathcal{N}},\varepsilon}\bigg(\frac{1}{g_N}\bigg) \tag{7.79}$$

The claimed upper bound follows from combining the estimates obtained in (7.78) and (7.79). $\square$

The following theorem is an extension of Theorem 7.2.14, when both variables are in the neighborhoods of different parabolic fixed points.

Theorem 7.2.16. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume and genus $g_N \geq 1$, and let $p, q \in \mathcal{P}_N$ be two parabolic fixed points with $p \neq q$. Then, with the notations as in 7.2.4, for any $N \in \mathcal{N}$ sufficiently large, $0 < \varepsilon < 1$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have the upper bound*

$$\sup_{\substack{z\in U_{N,\varepsilon}(p)\\ w\in U_{N,\varepsilon}(q)}} \bigg| g_{N,\mathrm{can}}(z,w) - \sum_{\gamma\in S_{\Gamma_N}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\bigg| =$$
$$O_{q_{\mathcal{N}},\varepsilon,\delta}\bigg(g_N\,|\mathcal{P}_N|\bigg(1+\frac{1}{\lambda_{N,1}}\bigg)\bigg).$$

Proof. From Theorem 6.2.8, for any $N \in \mathcal{N}$, $0 < \varepsilon < 1$, $\alpha \in (0, \lambda_{N,1})$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have

$$\sup_{\substack{z \in U_{N,\varepsilon}(p) \\ w \in U_{N,\varepsilon}(q)}} \Big| g_{N,\mathrm{can}}(z,w) - \sum_{\gamma \in S_{\Gamma_N}(\delta; z,w)} g_{\mathbb{H}}(z,\gamma w) \Big| \leq$$
$$A_{N,\varepsilon,\alpha,\delta} + \frac{2\pi d_{X_N}}{(\log\varepsilon)^2 \operatorname{vol}_{\mathrm{hyp}}(X_N)}.$$

The claimed upper bound follows from the same arguments as in Theorem 7.2.15. □

Corollary 7.2.17. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume and genus $g_N \geq 1$, and let $p, q \in \mathcal{P}_N$ be two parabolic fixed points with $p \neq q$. Then, with the notations as in 7.2.4, for any $N \in \mathcal{N}$ sufficiently large, $0 < \varepsilon < 1$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have the upper bound*

$$\big| g_{N,\mathrm{can}}(p,q) \big| = O_{q_{\mathcal{N}},\varepsilon,\delta}\bigg(g_N \, |\mathcal{P}_N| \bigg(1 + \frac{1}{\lambda_{N,1}} \bigg) \bigg).$$

Proof. From Corollary 6.2.9, for any $N \in \mathcal{N}$, $0 < \varepsilon < 1$, $\alpha \in (0, \lambda_{N,1})$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have

$$\big| g_{N,\mathrm{can}}(p,q) \big| \leq A_{N,\varepsilon,\alpha,\delta} + \frac{2\pi d_{X_N}}{(\log\varepsilon)^2 \operatorname{vol}_{\mathrm{hyp}}(X_N)}.$$

The claimed upper bound follows from the same arguments as in Theorem 7.2.15. □

Lemma 7.2.18. *For any $N \in \mathbb{N}_{>0}$, let X_N be any of the modular curves $Y_0(N)$, $Y_1(N)$, or $Y(N)$ having genus bigger than zero. Then, with the notations as in 7.2.4, there is a constant $c > 0$ satisfying $\lambda_{N,1} \geq c$.*

Proof. The proof of the lemma follows from extending the arguments from Lemma 5.9 in [11] to modular curves of genus bigger than zero. □

The following Corollary is an extension of Corollary 5.10 in [11] to admissible sequences of non-compact modular curves.

Corollary 7.2.19. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume and genus $g_N \geq 1$ of type (2), i.e., of modular curves. Then, with the notations as in 7.2.4, for any $N \in \mathcal{N}$ sufficiently large, $0 < \varepsilon < 1$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have the following upper bounds*

$$\sup_{z,w \in Y_{N,\varepsilon}} \big| g_{N,\mathrm{hyp}}(z,w) - g_{N,\mathrm{can}}(z,w) \big| = O_{q_{\mathcal{N}},\varepsilon,\delta}\big(g_N \, |\mathcal{P}_N| \big),$$
$$\sup_{z,w \in Y_{N,\varepsilon}} \Big| g_{N,\mathrm{can}}(z,w) - \sum_{\gamma \in S_{\Gamma_N}(\delta; z,w)} g_{\mathbb{H}}(z,\gamma w) \Big| = O_{q_{\mathcal{N}},\varepsilon,\delta}\big(g_N \, |\mathcal{P}_N| \big).$$

Proof. The proof of the corollary follows directly from combining the upper bounds obtained in Theorems 7.2.13 and 7.2.14 with the lower bound obtained for the first eigenvalue $\lambda_{N,1}$ of any of the modular curves $Y_0(N)$, $Y_1(N)$, or $Y(N)$ in Lemma 7.2.18. $\square$

Corollary 7.2.20. *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann surfaces of finite hyperbolic volume and genus $g_N \geq 1$ of type (2), i.e., of modular curves, and let $p, q \in \mathcal{P}_N$ be two parabolic fixed points with $p \neq q$. Then, with the notations as in 7.2.4, for any $N \in \mathcal{N}$ sufficiently large, $0 < \varepsilon < 1$, and $\delta \in (0, \widetilde{r}_{N,\varepsilon})$, we have the upper bound*

$$\big|g_{N,\mathrm{can}}(p,q)\big| = O_{q_{\mathcal{N}},\varepsilon,\delta}\big(g_N\,|\mathcal{P}_N|\big).$$

Proof. The proof of the corollary follows from combining the upper bound obtained in Corollary 7.2.17 with the lower bound obtained for the first eigenvalue $\lambda_{N,1}$ of any of the modular curves $Y_0(N)$, $Y_1(N)$, or $Y(N)$ in Lemma 7.2.18. $\square$

Remark 7.2.21. For $N \in \mathcal{N}$, the constant $c_{N,\varepsilon}$ associated to the Riemann surface X_N does not behave well in covers, which makes it difficult to estimate it in covers. So the estimates which we have derived above depend on the choice of a $\delta \in (0, \widetilde{r}_{N,\varepsilon})$. However, for a sufficiently small choice of ε, it can be shown that the constant $c_{N,\varepsilon}$ satisfies the upper bound

$$\frac{1}{c_{N,\varepsilon}} = O_\varepsilon(1).$$

So if we choose $\varepsilon > 0$ sufficiently small, the dependence of our estimates on the constant $\delta > 0$ can be removed.

Appendix A

Special functions and net of modular curves

A.1 Bessel functions

For $z, v \in \mathbb{C}$, the modified Bessel function of the first kind is a solution of the differential equation

$$z^2 \frac{\partial^2 f}{\partial z^2} + z \frac{\partial f}{\partial z} - (z^2 - v^2) f = 0, \tag{A.1}$$

and is given by the power series

$$I_v(z) = \sum_{k=0}^{\infty} \frac{1}{k!\,\Gamma(k+1+v)} \left(\frac{z}{2}\right)^{v+2k}.$$

If $v \in \mathbb{Z}$, then as a function in z, $I_v(z)$ is a holomorphic function on $\mathbb{C}$. If $v \notin \mathbb{Z}$, then as a function in z, $I_v(z)$ is singular along the negative x-axis, i.e., along the set $\{z = x + iy \in \mathbb{C} |\, x < 0,\, y = 0\}$. The parameter v is called the order of the Bessel function.

Both $I_v(z)$ and $I_{-v}(z)$ are linearly independent if and only if v is not an integer. If $v \in \mathbb{Z}$, we have the relation

$$I_v(z) = I_{-v}(z).$$

We set

$$K_v(z) = \frac{\pi\big(I_{-v}(z) - I_v(z)\big)}{2\sin(\pi v)}.$$

The functions $I_v(z)$ and $K_v(z)$ are two linearly independent solutions of the differential equation (A.1). The function $K_v(z)$ is called the modified Bessel function of the second kind.

The modified Bessel functions of the first kind satisfy the following recurrence relations

$$
\begin{aligned}
I_{v-1}(z) - I_{v+1}(z) &= \frac{2vI_v(z)}{z}, \qquad I_{v-1}(z) + I_{v+1}(z) = 2\frac{\partial I_v(z)}{\partial z},\\
\frac{\partial\big(z^v I_v(z)\big)}{\partial z} &= z^v I_{v-1}(z), \qquad \frac{\partial\big(z^{-v} I_v(z)\big)}{\partial z} = \frac{I_{v+1}(z)}{z^v}.
\end{aligned} \tag{A.2}
$$

The modified Bessel functions of the second kind $K_v(z)$ satisfy the above formulae with a negative sign on the right-hand side.

The modified Bessel functions of half order are elementary functions given by the following formulae

$$
I_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sinh z, \quad K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z}. \tag{A.3}
$$

Using the recurrence relations elucidated in equation (A.2), and the formulae for Bessel functions of half order given by equation (A.3), one can obtain elementary expressions for Bessel functions of any half-integer v.

For $y > 1 + |v|^2$, we have the following asymptotic behavior for the modified Bessel functions

$$
\begin{aligned}
I_v(y) &= \sqrt{\frac{1}{2\pi y}}e^{y}\left(1 + O\left(\frac{1+|v|^2}{y}\right)\right),\\
K_v(y) &= \sqrt{\frac{\pi}{2y}}e^{-y}\left(1 + O\left(\frac{1+|v|^2}{y}\right)\right).
\end{aligned}
$$

A.2 Whittaker functions

A function $f : \mathbb{H} \to \mathbb{C}$ with continuous partial derivatives of order 2 is an eigenfunction of the hyperbolic Laplacian Δ_{hyp} with eigenvalue $\lambda \in \mathbb{C}$ if

$$
(\Delta_{\text{hyp}} - \lambda)\, f = 0. \tag{A.4}
$$

The Whittaker function of first order $V_s(z)$ is an eigenfunction of the hyperbolic Laplacian Δ_{hyp} with eigenvalue $\lambda = s(1-s)$ given by the formula

$$
V_s(z) = 2\pi\sqrt{y}\, I_{s-1/2}(2\pi y)\, e^{2\pi i x}. \tag{A.5}
$$

The Whittaker function of second order $W_s(z)$ is an eigenfunction of the Laplacian with eigenvalue $\lambda = s(1-s)$ given by the formula

$$
W_s(z) = 2\sqrt{y}\, K_{s-1/2}(2\pi y)\, e^{2\pi i x}. \tag{A.6}
$$

The Whittaker functions $V_s(z)$ and $W_s(z)$ are linearly independent solutions of the differential equation (A.4) for $\lambda = s(1-s)$.

The Whittaker functions $V_s(z)$ and $W_s(z)$ are extended to the entire complex-plane by imposing the symmetry

$$V_s(z) = V_s(\overline{z}), \qquad W_s(z) = W_s(\overline{z}).$$

The Whittaker functions $V_s(z)$ and $W_s(z)$ exhibit distinct behavior at infinity, namely

$$V_s(z) = O\big(e^{2\pi i\overline{z}}\big), \qquad W_s(z) = O\big(e^{2\pi iz}\big),$$

as y approaches infinity.

A.3 Net of modular curves

For $N \in \mathbb{N}_{>0}$ and for the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$, let

$$\begin{aligned}Y_0(N) &= \Gamma_0(N)\backslash\mathbb{H},\\ Y_1(N) &= \Gamma_1(N)\backslash\mathbb{H},\\ Y(N) &= \Gamma(N)\backslash\mathbb{H}\end{aligned}$$

be the corresponding modular curves, respectively. In each of the three cases above, let $\mathcal{N} \subseteq \mathbb{N}$ be such that $Y_0(N)$, $Y_1(N)$, $Y(N)$ has no torsion points and has genus bigger than zero for $N \in \mathcal{N}$, respectively. We then consider here the families $\{X_N\}_{N\in\mathcal{N}}$ given by

$$\{Y_0(N)\}_{N\in\mathcal{N}},\ \ \{Y_1(N)\}_{N\in\mathcal{N}},\ \ \{Y(N)\}_{N\in\mathcal{N}}.$$

Denote by $q_{\mathcal{N}} \in \mathcal{N}$ the smallest prime in $\mathcal{N}$. For example, we can choose $q_{\mathcal{N}} = 11$.

The families of modular curves $\{X_N\}_{N\in\mathcal{N}}$ do not form a single tower of hyperbolic Riemann surfaces. However, they can be parametrized by a set of integers $\mathcal{B}(q_{\mathcal{N}})$, which we call a net. This was first studied in Section 5 of [12].

Definition A.3.1. Let $\mathbb{P}$ denote the set of primes.

(1) We call $N \in \mathcal{N}$ *base hyperbolic*, if there exists no proper divisor N' of N such that the genus $g_{N'}$ of the modular curve $X_{N'}$ is bigger than zero.

(2) Put

$$\mathcal{B}_1(q_{\mathcal{N}}) = \{N \text{ base hyperbolic} \,|\, N = q_1^{\alpha_1}\cdot\ldots\cdot q_k^{\alpha_k},\, q_j \leq q_{\mathcal{N}},\, j = 1,\ldots,k \in \mathbb{N}\}.$$

From the above definition, it follows that the set $\mathcal{B}_1(q_{\mathcal{N}})$ is finite.

(3) Put

$$\mathcal{B}_2(q_{\mathcal{N}}) = \{q \in \mathbb{P} \,|\, q > q_{\mathcal{N}}\}.$$

(4) Put

$$\mathcal{B}(q_{\mathcal{N}}) = \mathcal{B}_1(q_{\mathcal{N}}) \cup \mathcal{B}_2(q_{\mathcal{N}}).$$

The set $\mathcal{B}(q_{\mathcal{N}})$ is called the net for the families of modular curves $\{X_N\}_{N\in\mathcal{N}}$.

From the above definition, it is easy to see that for every $N \in \mathcal{N}$, there exists a $N'|N$ with $N' \in \mathcal{B}_1(q_{\mathcal{N}})$ or a $q|N$ with $q \in \mathcal{B}_2(q_{\mathcal{N}})$. This implies that for any $N \in \mathcal{N}$, there exists a $N' \in \mathcal{B}(q_{\mathcal{N}})$ such that X_N is a finite degree cover of $X_{N'}$. This is how the set $\mathcal{B}(q_{\mathcal{N}})$ parametrizes the families of modular curves $\{X_N\}_{N\in\mathcal{N}}$.

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Acknowledgments

First and foremost, I wish to thank my doctoral advisor Prof. Dr. Jürg Kramer, for introducing me to the fascinating area of automorphic forms and Arakelov theory. I am greatly indebted to him, not only for the amount of mathematics that I have learnt from him, but also for the great amount of time that he has invested in teaching me the art of mathematical writing. I would also like to thank him for providing me continuous financial support during the full course of my doctoral studies.

I am very grateful to Prof. Dr. Jay Jorgenson and Dr. Anna von Pippich for their constant support and encouragement. I would like to thank them for taking the time to discuss my work, and introducing me to new concepts in number theory.

I am also quite indebted to Dr. Robin de Jong and Prof. Dr. Bas Edixhoven who introduced me to Arakelov theory and also supervised my master thesis. I am also quite thankful for their helpful remarks and kind suggestions.

I would also like to thank Prof. Dr. Walter Gubler for his insightful courses on “Arakelov theory” and “Arithmetic Riemann-Roch” during his stay in Berlin.

I would also like to express my gratitude to Dr. Peter Bruin, Ananyo Dan, Giovanni De Gaetano, Dr. Fritz Hörmannn, Barbara Jung, Alexandre Krestiachine, Frank Lapp, Dr. Paul Larsen, Dr. Hartwig Mayer, Nicholas Schmidt, Dr. Nicola Tarasca, Dr. Filippo Viviani, Christian Wald, and Dr. Nahid Walji for many interesting mathematical discussions.

I would like to thank the DFG Research Training Group “Arithmetic and Geometry” for supporting my studies for the years 2007–2010.

I would also like to thank the secretary of the “Institut für Mathematik”, Humboldt-Universität zu Berlin, Ms. Marion Thomma for her invaluable help in sorting out all kinds of bureaucratic hassles.

I am greatly indebted to my parents, my sister Pani, and my wife Niharika for their constant moral support, encouragement, and understanding throughout my doctoral years.

Finally I would like to thank my school teacher Mr. N. Sri Ram Murthy, for nurturing my interest in mathematics, and for his encouragement to take up mathematics as a career.